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# Dimensional Crossover and Thermo-Optic Interaction in Photon Bose-Einstein Condensates

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Seminar Talk: AK Pelster

15.01.24 and 17.01.24



# Topics

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## Part I

- Experimental Realization
- Theoretical Approach
  - Homogeneous Condensate
  - Trapped Condensate
- Temperature Diffusion

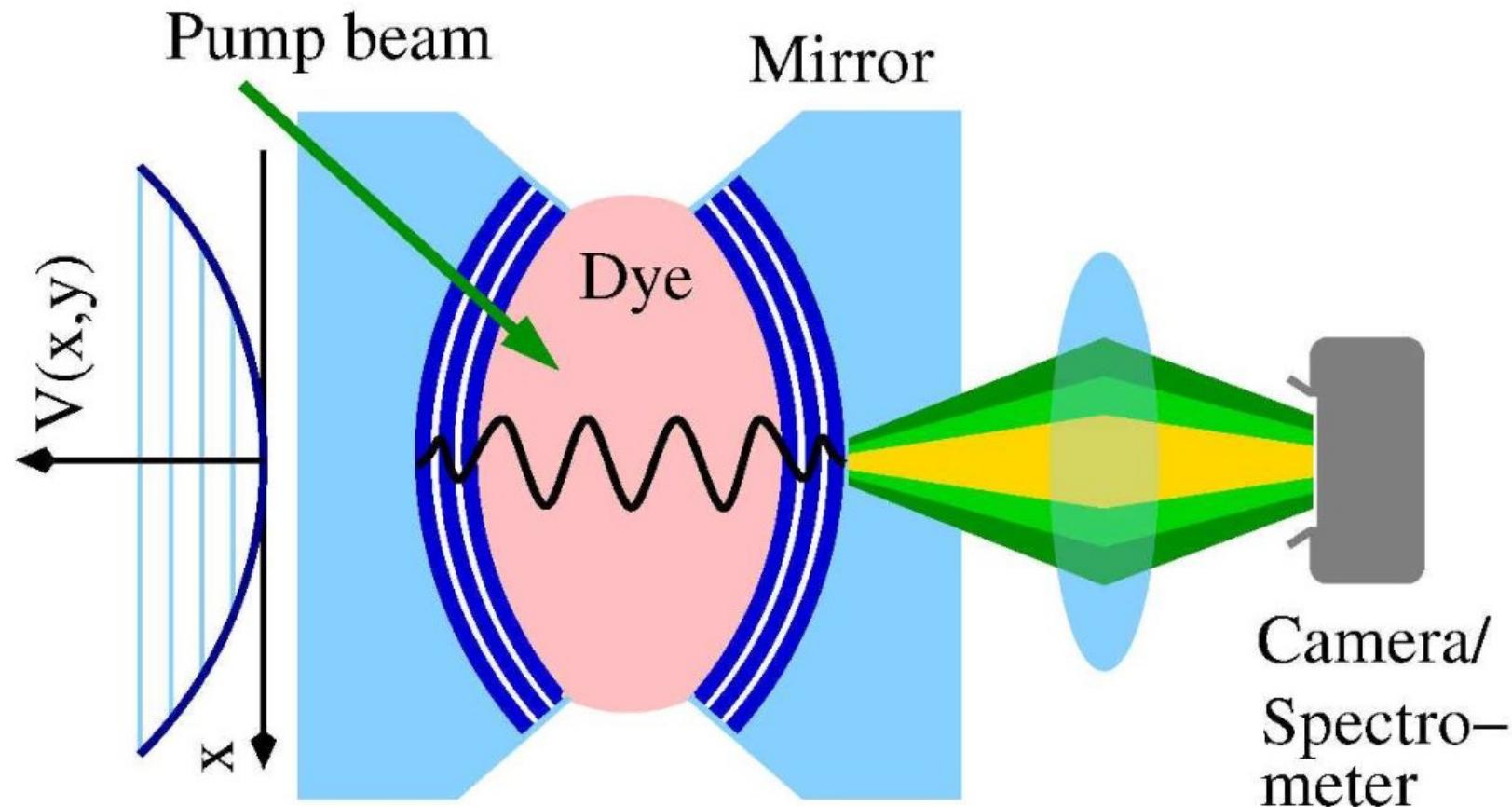
## Part II

- Dimensional Crossover 2D → 1D
  - Thermodynamics
  - Thermo-Optic Interaction
- Hartree-Fock Analogue of Thermo-Optic interaction
- Exact Diagonalisation of PBEC

E. Stein, *Open-Dissipative Mean-Field Theory for Photon Bose-Einstein Condensates*, Diploma thesis (2018)

E. Stein, *Dimensional Crossover and Thermo-Optic Interaction in Photon Bose-Einstein Condensates*, PhD thesis (2022)

# Experimental Realization - Setup



J. Klaers, et al., Nature **468**, 545 (2010)

# Experimental Realization - Cavity

- Paraxial approximation

$$E = \hbar c |\mathbf{k}| = \hbar c \sqrt{k_z^2 + k_r^2} \approx mc^2 + \frac{\hbar^2 \mathbf{k}_r^2}{2m} + \frac{m\Omega^2}{2} r^2$$

- Mapping to massive bosons in 2D harmonic trap

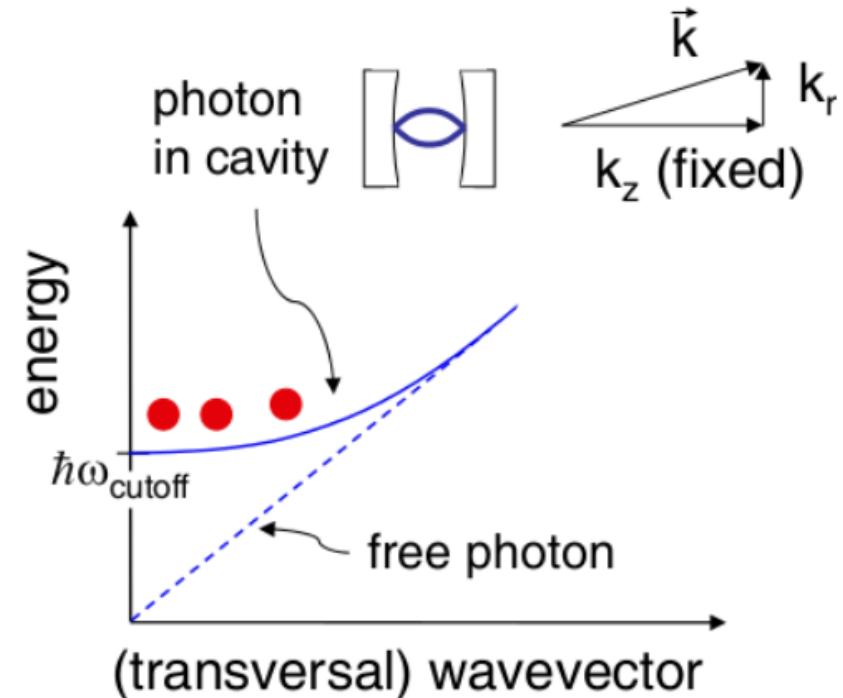
- Mass

$$m = \frac{7\hbar\pi n_0}{L_0 c^2} \approx 7 \times 10^{-36} \text{kg} \approx 10^{-10} m_{Rb}$$

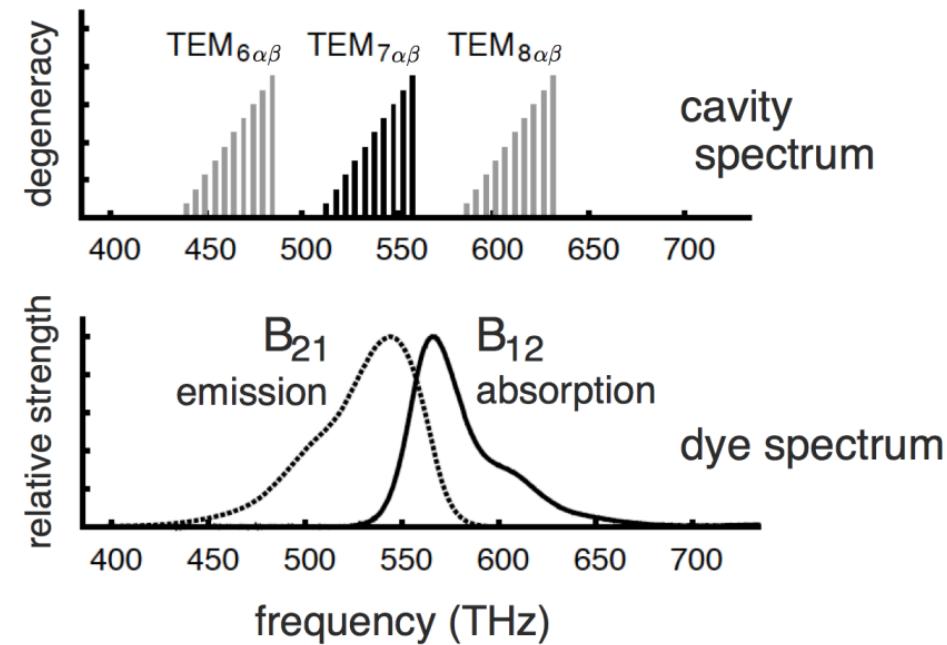
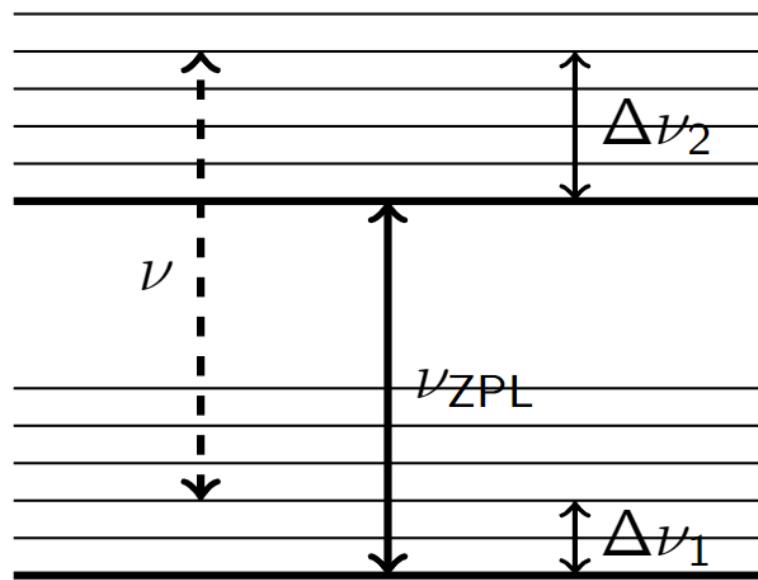
- Trap frequency

$$\Omega = \frac{c}{n_0} \sqrt{\frac{2}{L_0 R}} \approx 2 \times 10^{11} \text{Hz}$$

- Refraction index:  $n_0 = 1.46$  (Rhodamin 6G)
- Unperturbed length:  $L_0 = 1.5 \times 10^{-6} \text{m}$
- Mirror curvature:  $R = 1 \text{m}$



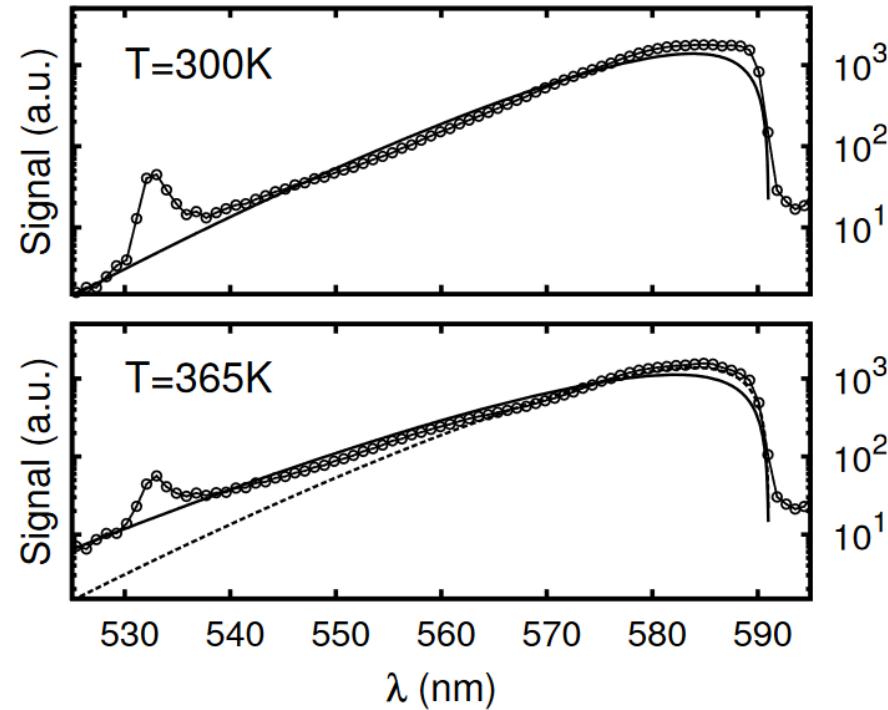
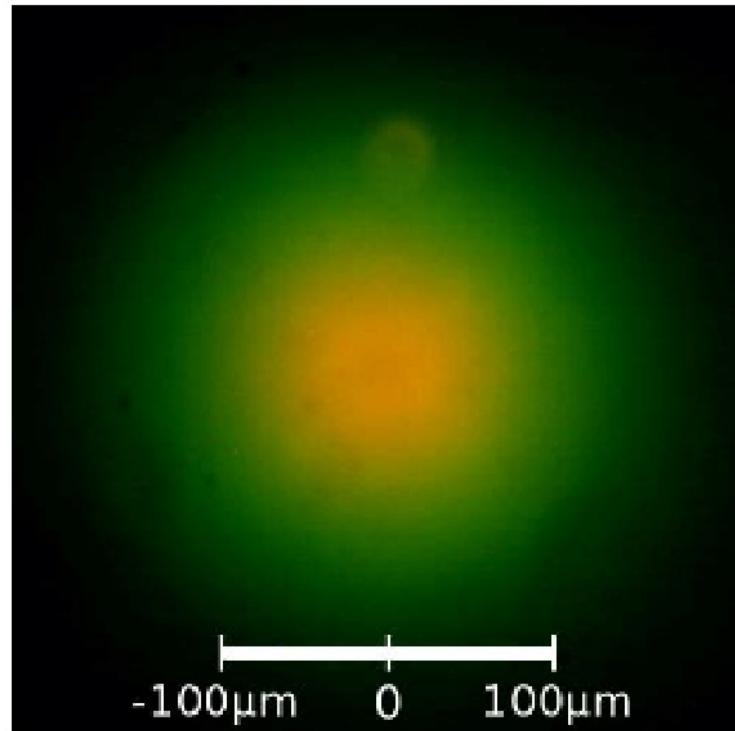
# Experimental Realization - Dye



- Kennard-Stepanov relation:

$$\frac{B_{21}(\nu)}{B_{12}(\nu)} = \exp \left[ -\frac{\hbar(\nu - \nu_{ZPL})}{k_B T} \right]$$

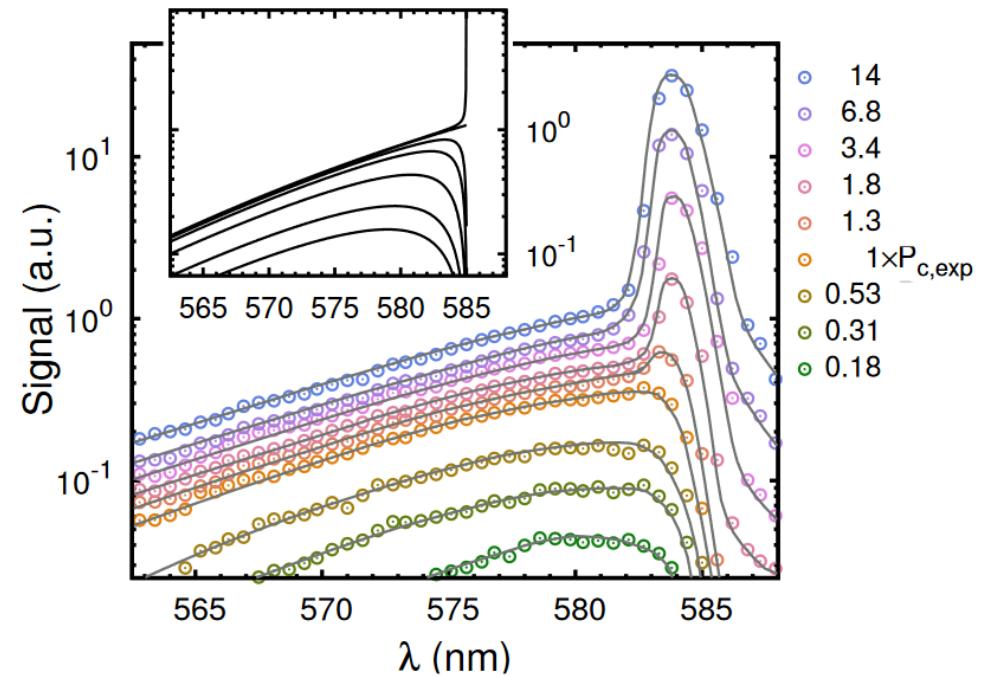
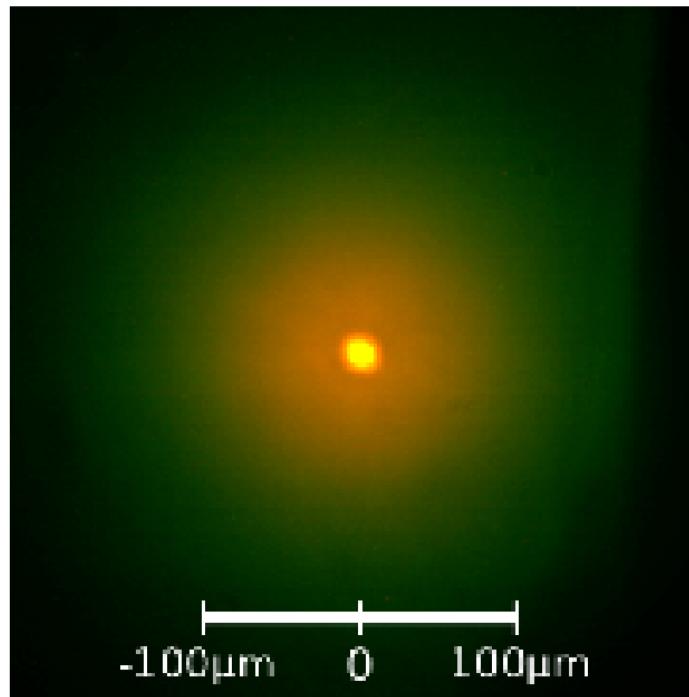
# Experimental Realization - Thermalisation



J. Klaers, et al., Nature **468**, 545 (2010)

- Thermalisation to  $T_{spec} \approx T_{room}$   $N < N_c = \frac{\pi^2}{3} \left( \frac{k_B T}{\hbar \Omega} \right)^2$
- See also:
  1. A. Pelster, Physik-Journal 10, Nr. 1, 20 (2011)
  2. A. Pelster, Physik-Journal 13, Nr. 3, 20 (2014)

# Experimental Realization - Condensation

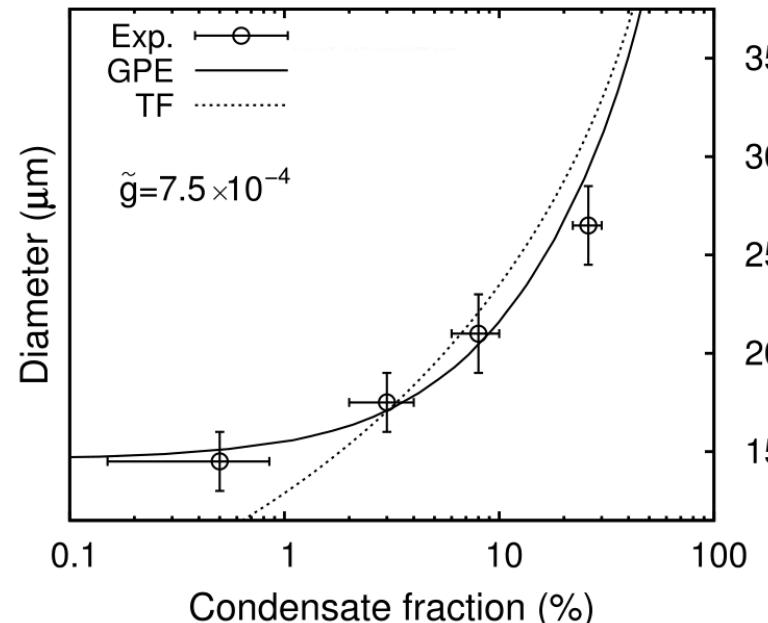


J. Klaers, et al., Nature **468**, 545 (2010)

- Condensation to  $T_{\text{spec}} \approx T_{\text{room}}$  @  $N_c = \frac{\pi^2}{3} \left( \frac{k_B T}{\hbar \Omega} \right)^2 \approx 7.7 \times 10^4$

$$N_c^{\text{exp}} = (6.3 \pm 2.4) \times 10^4$$

# Experimental Realization - Interaction



J. Klaers, et al., Nature **468**, 545 (2010)

- Gross-Pitaevskii equation:

$$i\hbar\partial_t\psi(\mathbf{r}, t) = -\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r}, t) + \frac{m\Omega^2}{2}r^2\psi(\mathbf{r}, t) + g|\psi(\mathbf{r}, t)|^2\psi(\mathbf{r}, t)$$

- Thomas-Fermi width:

$$q_0^{\text{TF}} = 2 \sqrt{\frac{\hbar}{\sqrt{\pi} m\Omega}} (\tilde{g} N_0)^{1/4}$$

J. Klaers, et al., Appl. Phys. B **105**, 1, 17-33 (2011)

- Dimensionless interaction constant  $\tilde{g} = \frac{gm}{\hbar^2}$
- Bonn  $\tilde{g} = 7.5 \times 10^{-4}$

# Experimental Realization - Interaction

## Photon-Photon Interaction Mechanisms

- Spatially/temporally varying refractive index leads to photon potential  $\rightarrow$  Photon-photon interaction
- Two contributions:
  1. Kerr effect:
    - Change of index of refraction due to light intensity
    - Instantaneous interaction
  2. Thermal lensing:
    - Heating of dye
    - Change of index of refraction due to temperature
    - Non-local in space and time

$$n = n_0 + n_2 I(\mathbf{r})$$

Thermo-optic coefficient  $\frac{\partial n}{\partial T} \approx -4.7 \times 10^{-4} \frac{1}{K}$

$$n = n_0 + \frac{\partial n}{\partial T} T(\mathbf{r}, t)$$

– **Main contribution**



Correction in wave equation:  $-\frac{mc^2}{n_0^2} \frac{\partial n}{n_0 \partial T} T \psi$

# Theoretical Approach

## Photon wave function:

- Starting point:

Refraction index   Extinction coefficient

$$\left( \nabla^2 - \frac{n^2 + 2i\gamma}{c^2} \partial_t^2 \right) \psi = 0$$

Electric field

3D wave equation

Vacuum light velocity

- Paraxial approximation

→ Light propagates close to optical axis

Photons behave like massive bosons trapped in harmonic oscillator.

- Slowly varying amplitude

$$\omega_{\text{cutoff}} = \frac{mc^2}{\hbar}$$

$$\psi(\mathbf{r}, z, t) = \psi(\mathbf{r}, t) \sin(k_z(\mathbf{r})z) e^{-i\omega_{\text{cutoff}}t}$$

$$k_z(r) = \frac{7\pi}{L_0} \left[ 1 + \frac{r^2}{L_0 R} \right]$$

$$m = \frac{7\hbar\pi n_0}{L_0 c} \sim 1 \times 10^{-36} \text{ kg}$$

$$\Omega = \frac{c}{n_0} \sqrt{\frac{2}{L_0 R}} \sim 1 \times 10^{11} \text{ Hz}$$

- Add pump and temperature corrections

Conversion factor    $N = \frac{n_0^2}{mc^2}$    pump power  $p$

$$i\hbar \partial_t \psi = \left[ -\frac{\hbar^2}{2m} \nabla^2 + \frac{m\Omega^2}{2} r^2 + g_T T \right] \psi + \frac{i\hbar}{2} \left[ Np - \Gamma + 2 \frac{\partial n}{n_0 \partial T} (Np + \Gamma) T \right] \psi$$

Photon-temperature coupling  $g_T = -mc^2 \frac{\partial n}{n_0^2 \partial T}$

Loss rate  $\Gamma$

# Theoretical Approach

## Temperature in Microcavity:

- Diffusion equation:

$$\partial_t T = D \nabla^2 T + B |\psi|^2$$

Thermal diffusion coefficient

3D heating coefficient

- Dimensional reduction:

$$T(\mathbf{r}, z, t) = \sum_v T_v(\mathbf{r}, t) \sin(k_v z)$$
$$k_v = \frac{v\pi}{L_0}$$
$$\tau = \frac{L_1^2}{4\pi^2 D_1} \text{ temperature relaxation time}$$
$$\rightarrow \boxed{\partial_t T = \left( \frac{-1}{\tau} + D \nabla^2 \right) T + B |\psi|^2}$$

2D diffusion equation

Exact solution:

$$T(\mathbf{r}, t) = B \int_{-\infty}^t dt' \int d^2 r' G(\mathbf{r} - \mathbf{r}', t - t') |\psi(\mathbf{r}', t')|^2$$

$$G(\mathbf{r}, t) = \frac{1}{4\pi D t} e^{-\frac{r^2}{4Dt} - \frac{t}{\tau}}$$

# Theoretical Approach

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Summary of Model:

$$i\hbar\partial_t\psi = \left[ -\frac{\hbar^2}{2m}\nabla^2 + \frac{m\Omega^2}{2}r^2 + g_T T \right]\psi + \frac{i\hbar}{2} \left[ Np - \Gamma + 2\frac{\partial n}{n_0 \partial T} (Np + \Gamma)T \right] \psi$$

$$\partial_t T = \left( \frac{-1}{\tau} + D\nabla^2 \right) T + B|\psi|^2$$

Reduces to one equation with       $T(\mathbf{r}, t) = B \int_{-\infty}^t dt' \int d^2r' G(\mathbf{r} - \mathbf{r}', t - t') |\psi(\mathbf{r}', t')|^2$

- Paraxial approximation: Photons in microcavity behave like massive bosons in harmonic trap
- Open-dissipative system: Pump and dissipation explicitly part of dynamics
- Photon-photon interaction: Due to change of temperature

# Application: Homogeneous System

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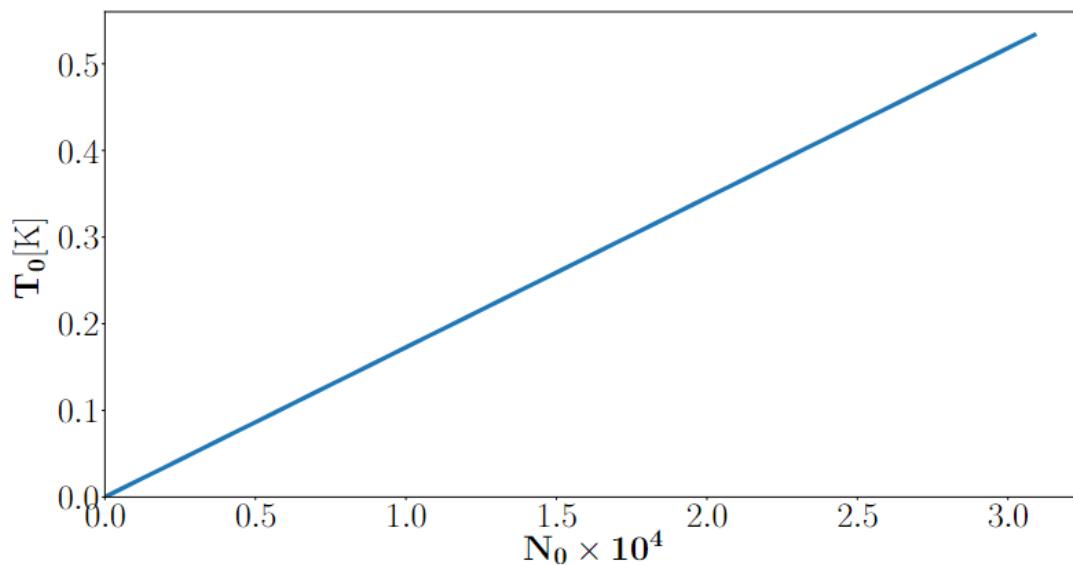
$$\begin{aligned} & i\hbar\partial_t\psi \\ &= \left[ -\frac{\hbar^2}{2m}\nabla^2 + g_T B \int_{-\infty}^t dt' \int d^2r' G(\mathbf{r} - \mathbf{r}', t - t') |\psi(\mathbf{r}', t')|^2 \right] \psi \\ &+ \frac{i\hbar}{2} \left[ Np - \Gamma + 2 \frac{\partial n}{n_0 \partial T} (Np + \Gamma) B \int_{-\infty}^t dt' \int d^2r' G(\mathbf{r} - \mathbf{r}', t - t') |\psi(\mathbf{r}', t')|^2 \right] \psi \end{aligned}$$

Bogoliubov-like ansatz for  
stability analysis:

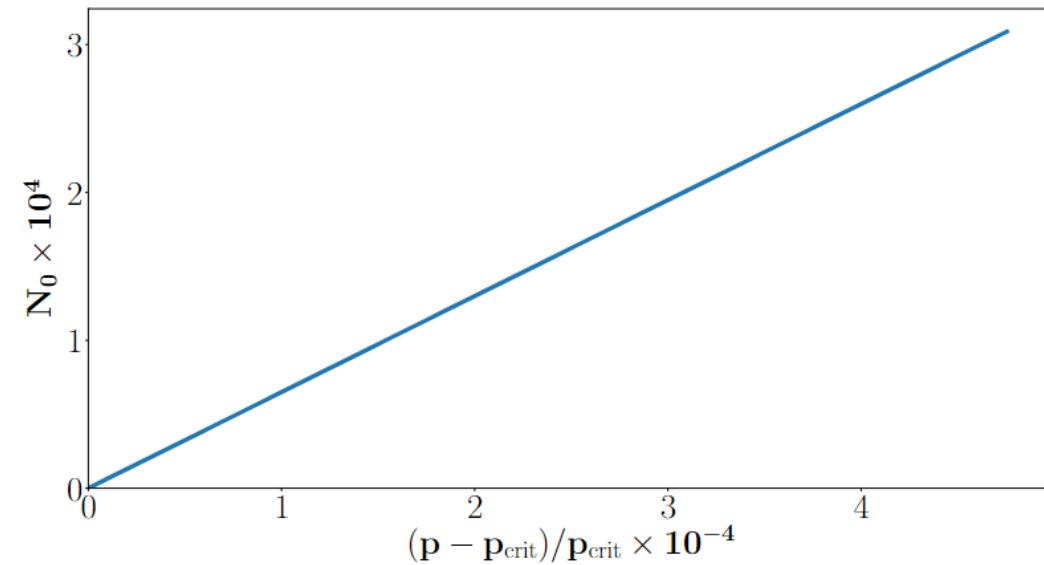
$$\psi(\mathbf{r}, t) = \psi_0 e^{-i\frac{\mu t}{\hbar}} [1 + u_k(\mathbf{r}, t) e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} + v_k^*(\mathbf{r}, t) e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega^* t)}]$$

# Application: Homogeneous System

Steady state:



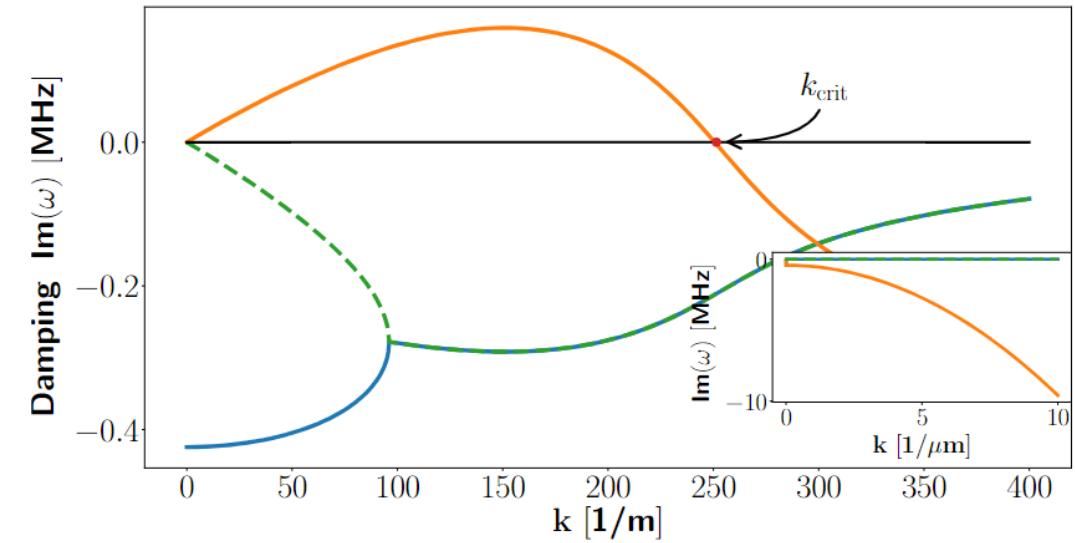
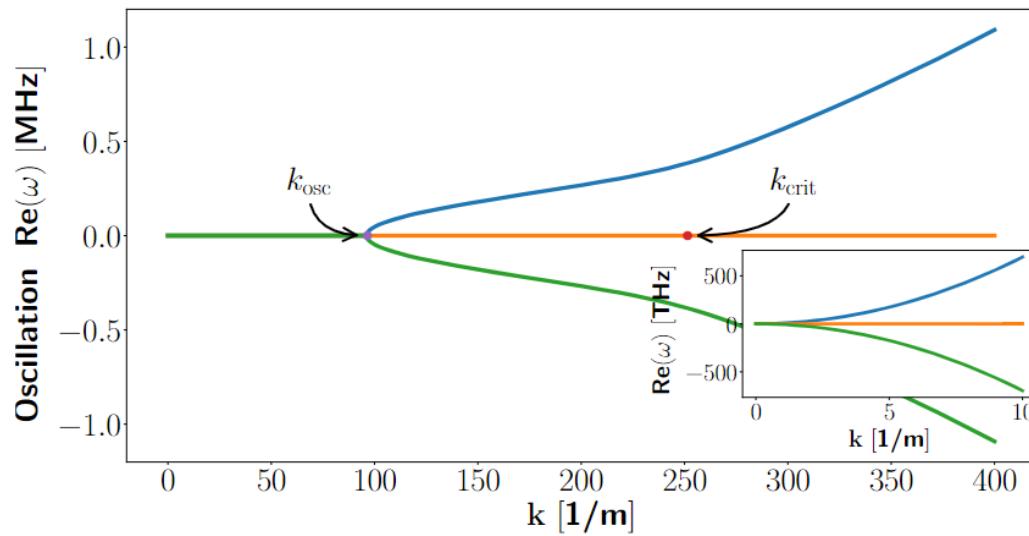
→ Linear increase



→ Photon number increases rapidly

# Application: Homogeneous System

## Dynamical Stability:



Goldstone theorem still valid,  
although Mermin-Wagner-  
Hohenberg theorem is not  
valid

15

→ Unstable condensate for  
small wave vectors due  
to large damping

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# Application: Trapped System

$$\begin{aligned} & i\hbar\partial_t\psi \\ &= \left[ -\frac{\hbar^2}{2m}\nabla^2 + \frac{m\Omega^2}{2}r^2 + g_T B \int_{-\infty}^t dt' \int d^2r' G(\mathbf{r} - \mathbf{r}', t - t') |\psi(\mathbf{r}', t')|^2 \right] \psi \\ &+ \frac{i\hbar}{2} \left[ Np - \Gamma + 2 \frac{\partial n}{n_0 \partial T} (Np + \Gamma) B \int_{-\infty}^t dt' \int d^2r' G(\mathbf{r} - \mathbf{r}', t - t') |\psi(\mathbf{r}', t')|^2 \right] \psi \end{aligned}$$

- Problem for analysis: no action available
- Idea: Use properties of Gaussian ansatz function

→ Project cumulants from  
equation of motion → • Centre-of-mass  
• width

Ansatz:

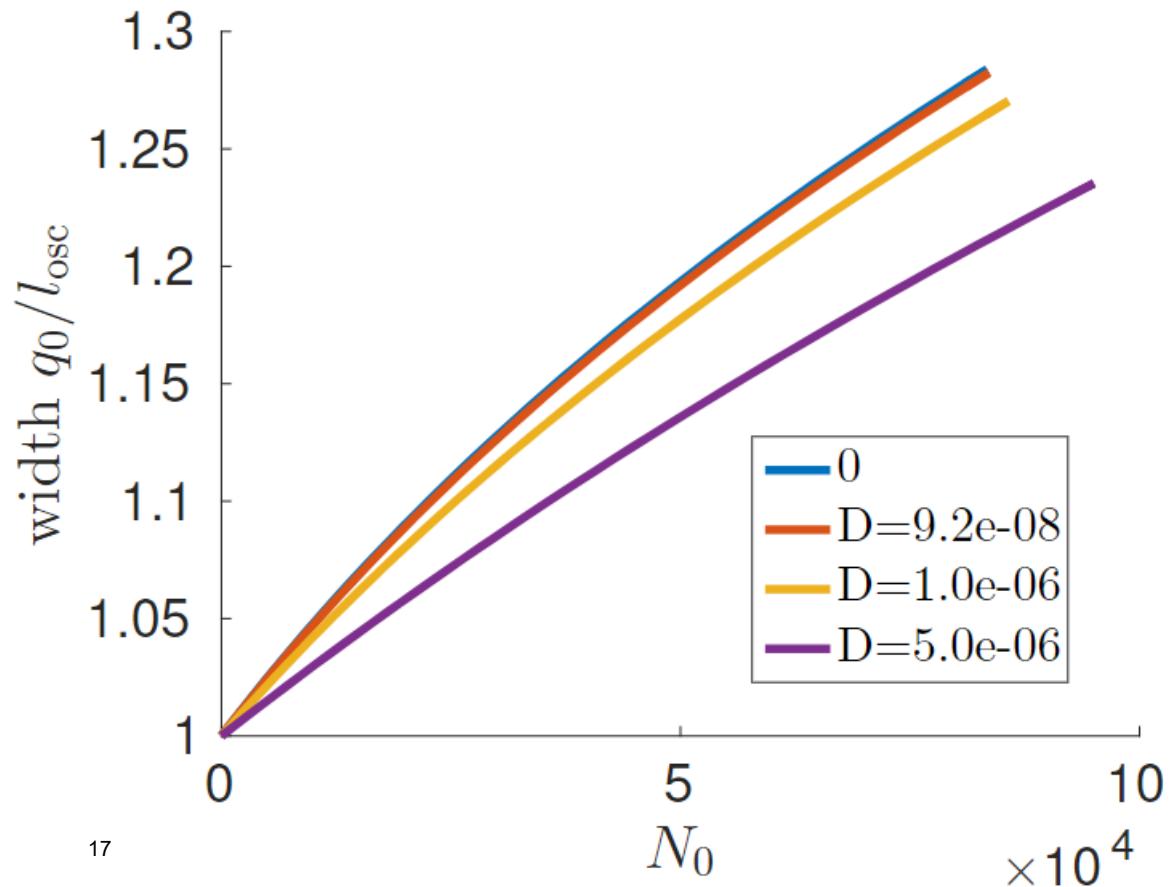
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$$\psi(x, t) = \sqrt{\frac{N(t)}{\pi q_1(t)q_2(t)}} \exp \left[ \sum_{j=1}^2 - \left( \frac{1}{2q_j(t)} + iA_j(t) \right) (x_j - x_{0j}(t))^2 + ix_j C_j(t) \right]$$

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# Application: Trapped System

Equilibrium:



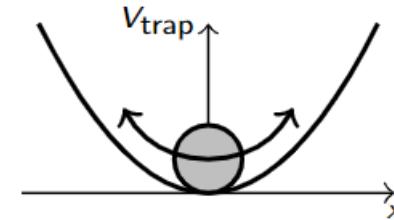
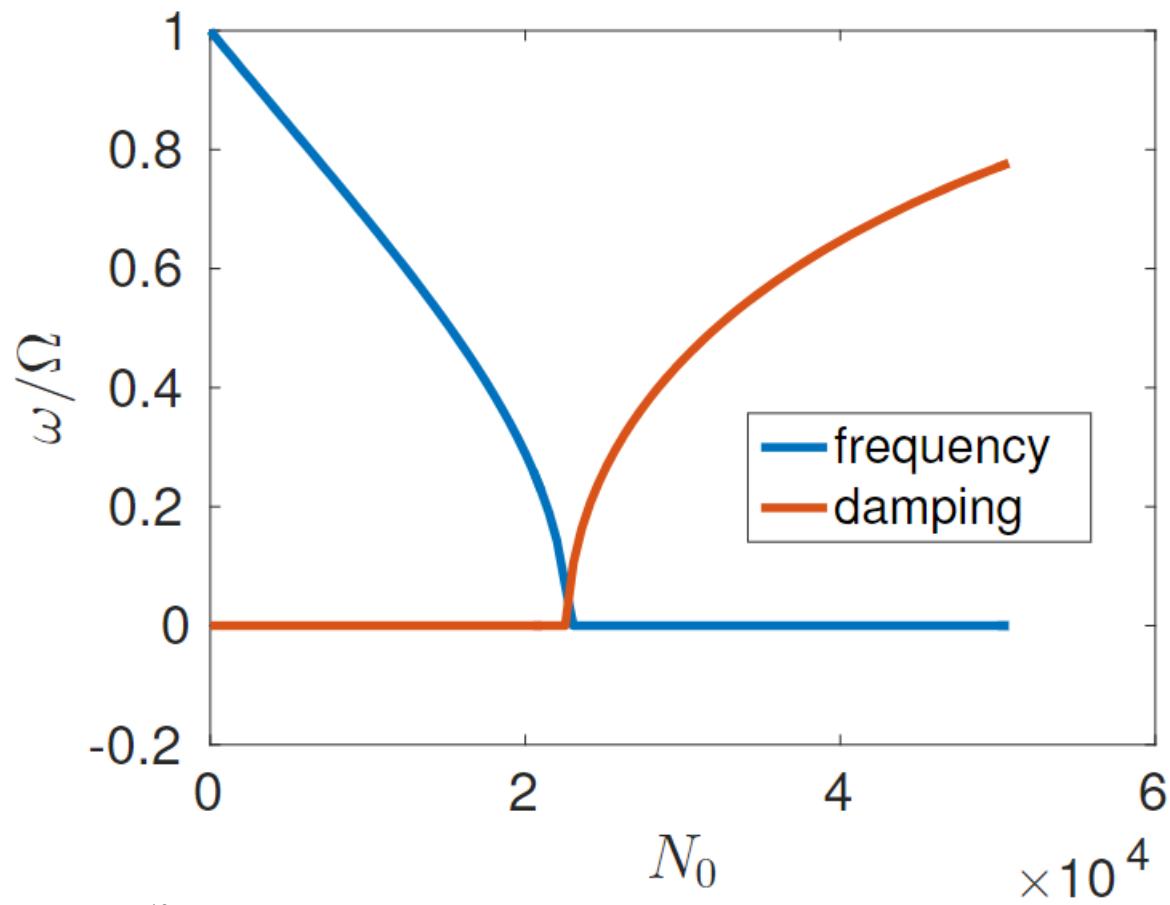
- For vanishing diffusion:

$$q_0 = l_{osc} \left( 1 + \frac{N_0 \tilde{g}}{2\pi} \right)^{1/4}$$

- $l_{osc} \approx 1 \times 10^{-6}\text{m}$   
→  $\tilde{g} \approx 1.29 \times 10^{-4}$
- Neglecting diffusion leads to systematic error in experimental analysis  
→ Error of 2%

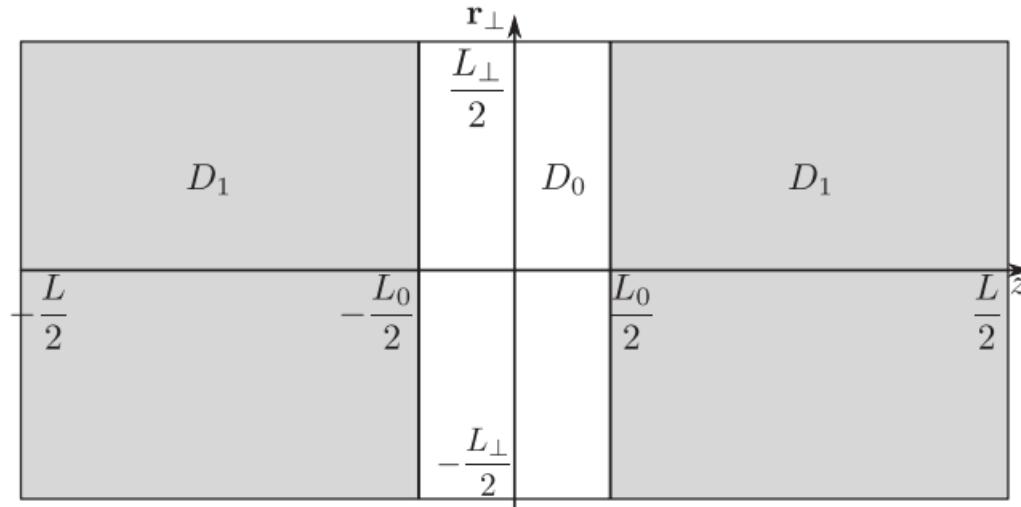
# Application: Trapped System

Dipole Mode:



- Instantaneous interaction:  
Kohn theorem:  $\omega = \Omega$
- Breakdown of Kohn theorem  
due to retardation and  
dipole frequency vanishes  
for certain photon number

# Temperature Diffusion



E. Stein, et al., NJP **21**, 103044 (2019)

- Temperature diffusion not negligible:

$$\partial_t T = [\partial_z(D(z)\partial_z) + D(z)\nabla^2]T + \text{Source}$$

- Boundary conditions:

$$\left. \partial_z T \right|_{z=0} = 0$$



$$\frac{1}{\tau} = \frac{4\pi^2 D_1}{L_{\perp}^2} + \frac{\pi D_1}{L_1^2}$$

$$T\left(z = \frac{L}{2}\right) = 0$$

$$L_1 = L - L_0$$

Connection to interaction:  $\tilde{g}_T \propto \tau$

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# Topics

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## Part I

- Experimental Realization
- Theoretical Approach
  - Homogeneous Condensate
  - Trapped Condensate
- Temperature Diffusion

## Part II

- Dimensional Crossover  $2D \rightarrow 1D$ 
  - Thermodynamics
  - Thermo-Optic Interaction
- Hartree-Fock Analogue of Thermo-Optic interaction
- Exact Diagonalisation of PBEC

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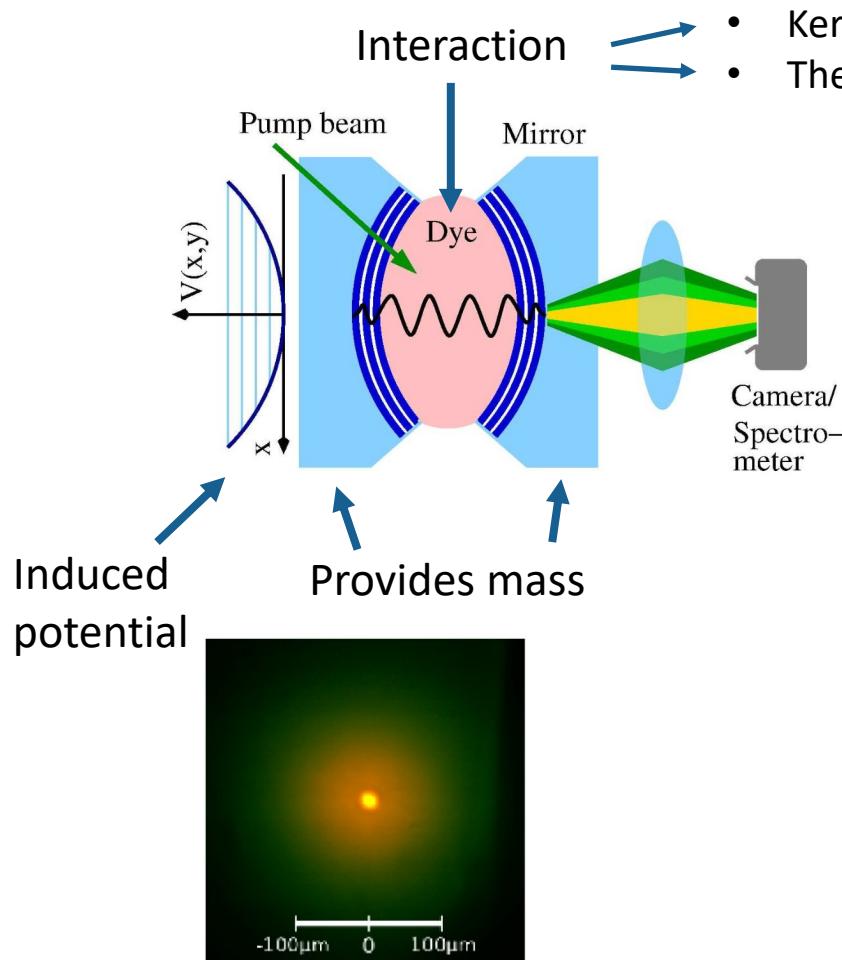
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# Summary of Part I

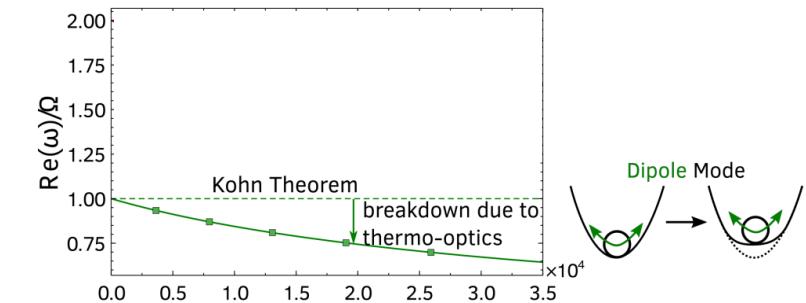
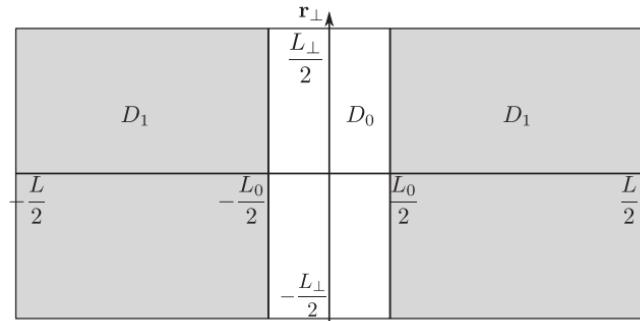


- Interaction  $\rightleftharpoons$
- Kerr effect
- Thermal lensing

**Theoretical 2D mean-field model:**

$$\partial_t T = \left( \frac{-1}{\tau} + D \nabla^2 \right) T + B |\psi|^2$$

$$i\hbar \partial_t \psi = \left[ -\frac{\hbar^2}{2m} \nabla^2 + \frac{m\Omega^2}{2} r^2 + g_T T \right] \psi + \frac{i\hbar}{2} \left[ Np - \Gamma + 2 \frac{\partial n}{n_0 \partial T} (Np + \Gamma) T \right] \psi$$



$$\rightarrow \frac{1}{\tau} = \frac{4\pi^2 D_1}{L_{\perp}^2} + \frac{\pi D_1}{L_z^2}$$

$$T_{\text{spec}} \approx T_{\text{room}} @ N_c = \frac{\pi^2}{3} \left( \frac{k_B T}{\hbar \Omega} \right)^2 \approx 7.7 \times 10^4$$

$\rightarrow$  Connection to interaction:  $\tilde{g}_T \propto \tau$

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# Dimensional Crossover: Thermodynamics

2D harmonic trap:

$$E_{jn}(\lambda) = \hbar\Omega \left( j + \lambda n + \frac{1 + \lambda}{2} \right)$$

Trap-aspect ratio  $\lambda = \frac{\Omega_y}{\Omega}$

Effective 1D, if energy spacing in y-direction  
larger than thermal energy:

$$\lambda > \lambda_{1D} = \frac{k_B T}{\hbar\Omega} \approx 160$$

Grand-canonical potential:

$$\Pi = -\frac{g}{\beta} \sum_{j,n=0}^{\infty} \sum_{k=1}^{\infty} \frac{e^{-\beta(E_{jn}(\lambda)-\mu)k}}{k} = \Pi_{1D} + \Delta\Pi(\lambda)$$

$\beta = \frac{1}{k_B T}$

# Dimensional Crossover: Thermodynamics

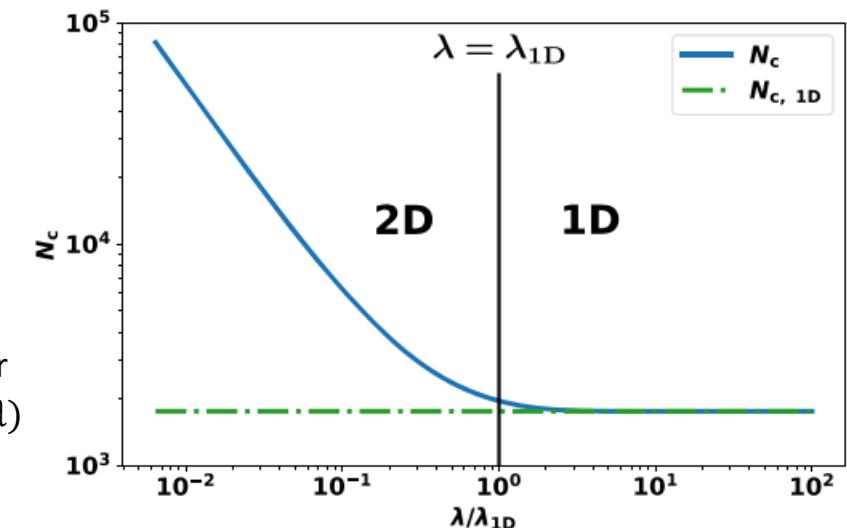
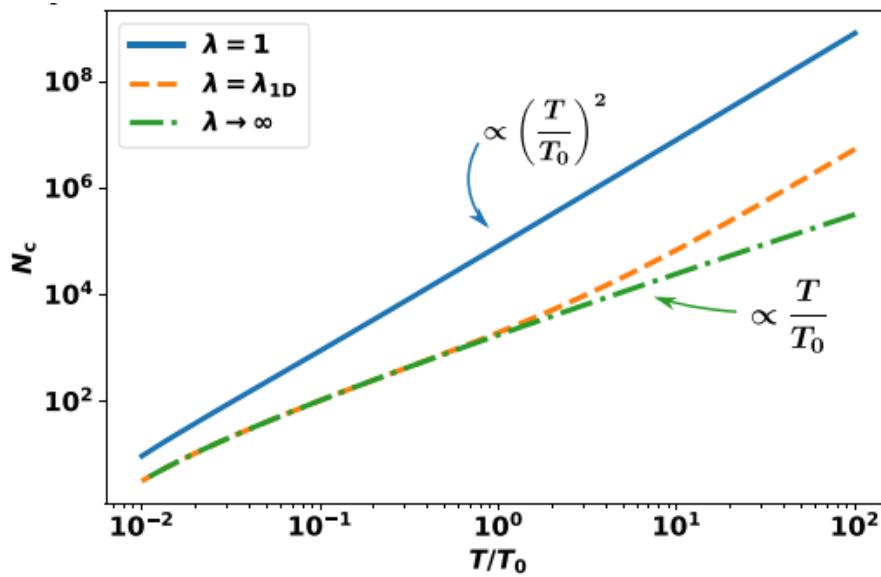
Particle number:  $N = -\frac{\partial \Pi}{\partial \mu}$

→ Critical particle number:  $N \approx N_0 + N_c$

$N_0 = \frac{g}{e^{\tilde{\mu}b} - 1} \approx \frac{g}{\tilde{\mu}b}$

Deep condensate limit

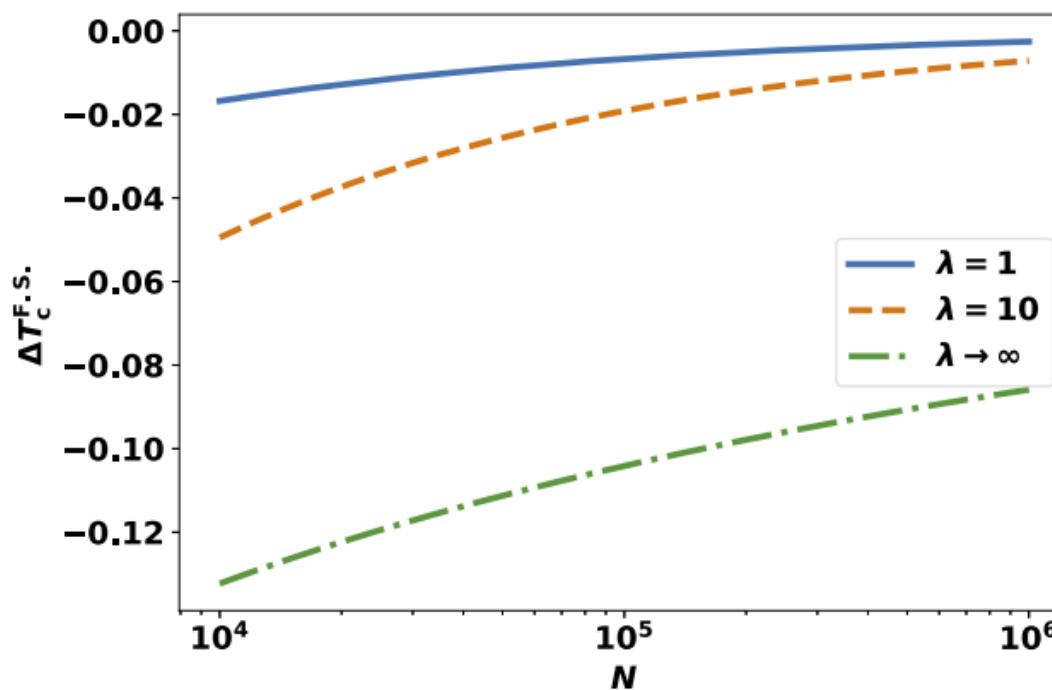
Critical particle number  
 $N_c = N_c^{1D} + \Delta N_c(\lambda)$



$$N_c^{1D} = g \frac{k_B T}{\hbar \Omega} \left[ \gamma - \ln \left( \frac{\hbar \Omega}{k_B T} \right) \right]$$

# Dimensional Crossover: Thermodynamics

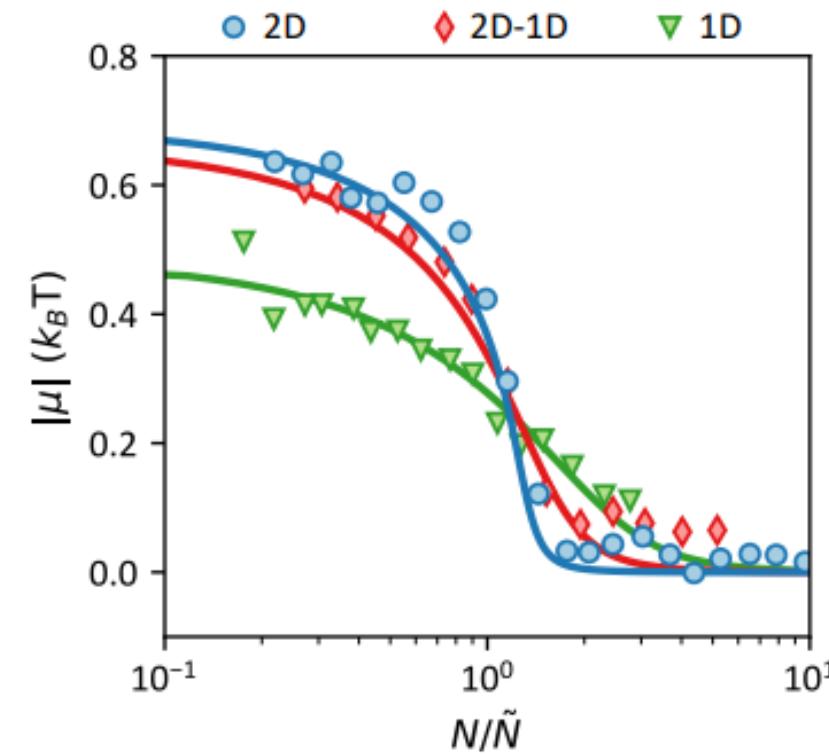
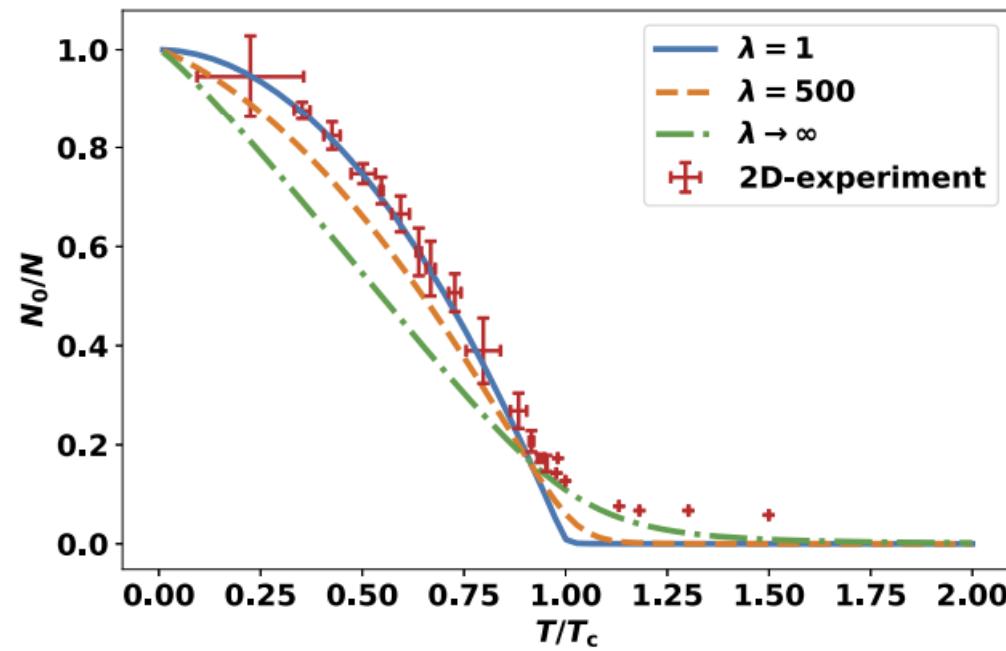
- Critical Temperature via  $N_c(T_c) = N \xrightarrow{1D} T_{c,1D} = \frac{\hbar\Omega}{gk_B} \frac{N}{\gamma - \ln\left(\frac{\hbar\Omega}{k_B T_{c,1D}}\right)}$
- Finite size corrections  $\Delta T_{c,\cdot} = \frac{T_{c,\cdot} - T_{c,\cdot}^{L.T.}}{T_{c,\cdot}^{L.T.}}$  ← Leading term for large  $N$



# Dimensional Crossover: Thermodynamics

- Condensate fraction

$$\frac{N_0}{N} \approx 1 - \frac{N_c}{N}$$

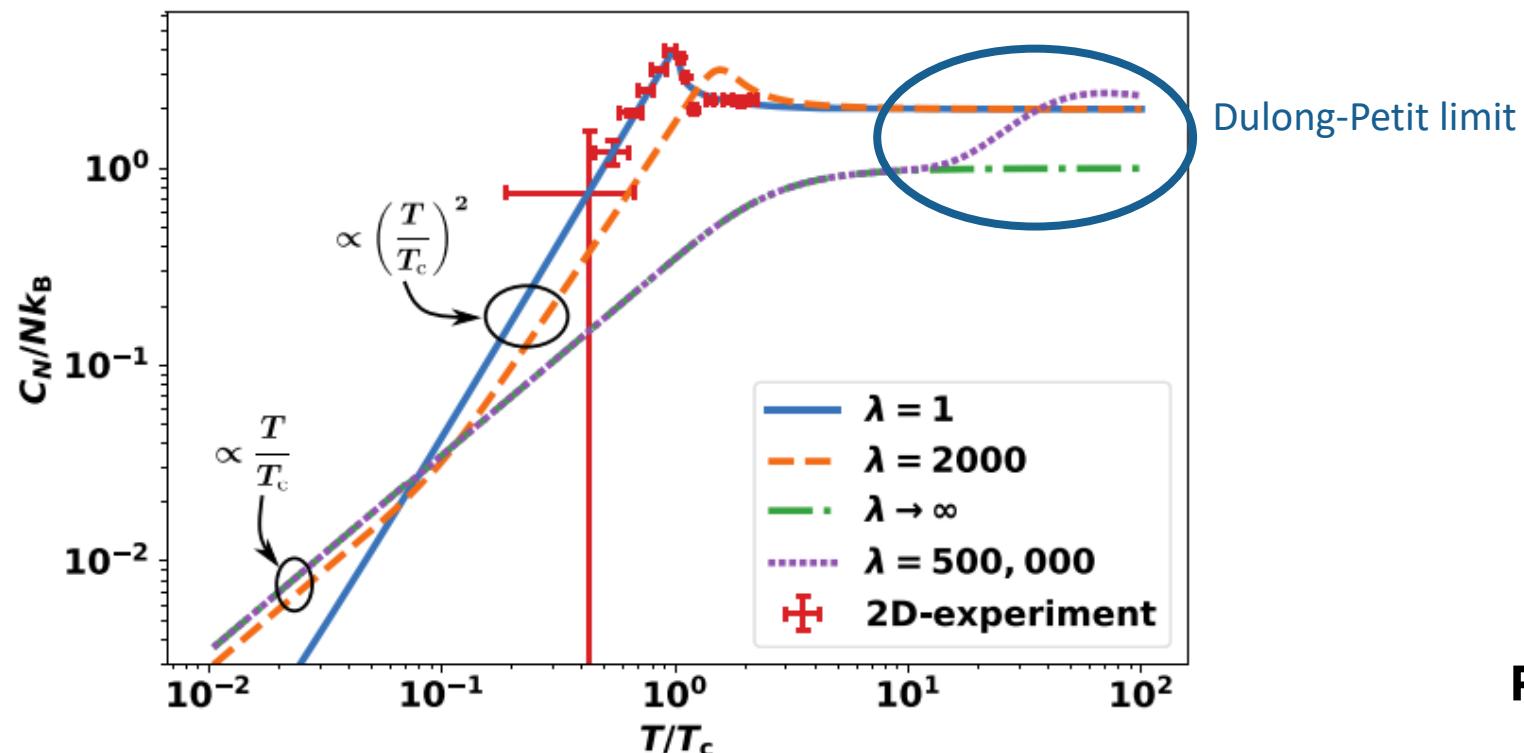


K. K. Umesh, et al., arXiv: 2311.10485v1 (2023)

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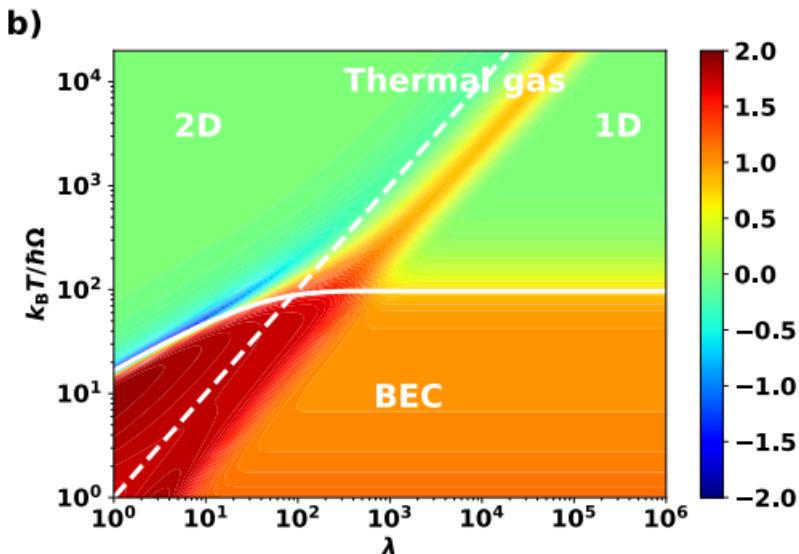
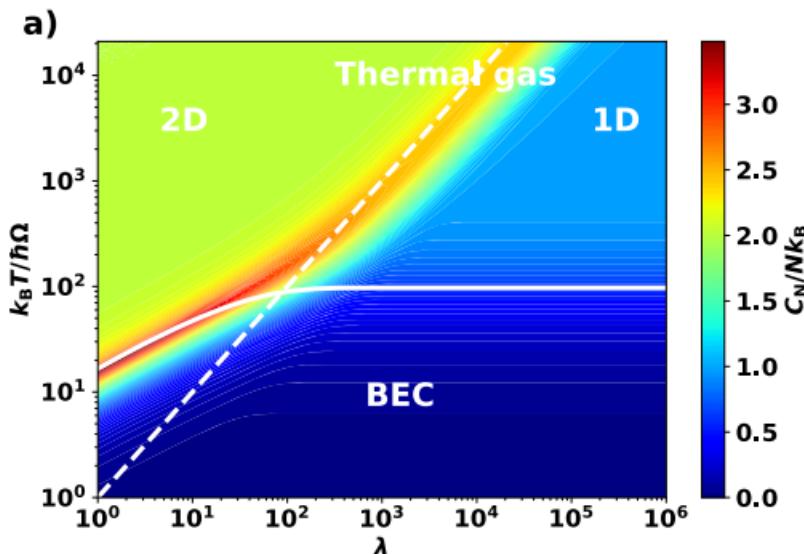
# Dimensional Crossover: Thermodynamics

- Specific heat:  $c_N = \frac{\partial U}{\partial T} + \frac{\partial U}{\partial \mu} \left( \frac{\partial \mu}{\partial T} \right)_N$
- Internal energy:  $U = \Pi + TS + \mu N$



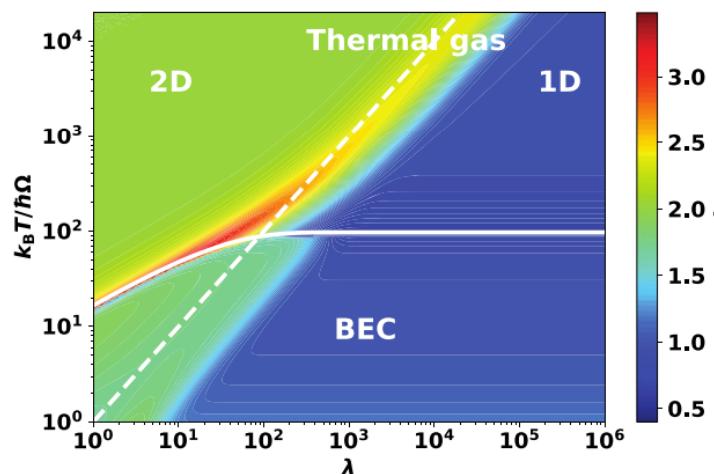
# Dimensional Crossover: Thermodynamics

## Phase diagram and effective dimension



Effective dimension in BEC phase:

$$d_{BEC} = -\frac{1}{Nk_B} \frac{\partial \ln(c_N)}{\partial \ln(b)}$$



# Dimensional Crossover: Thermo-Optic Interaction

Steady-State Model:

$$\mu\psi = \left( -\frac{\hbar^2 \nabla^2}{2m} + V + g_K |\psi|^2 + \gamma \Delta T \right) \psi$$

Kerr interaction

$$\Delta T = T - T_0 \rightarrow \Delta T = \tau D \nabla^2 \Delta T + \sigma \tau B |\psi|^2$$

Diffusion coefficient      Heating rate

Longitudinal relaxation      Duty cycle

Elimination of temperature difference

$$\Delta T(x) = \sigma \tau B \int_0^\infty dt \int d^2x' G(x - x', t) |\psi(x')|^2$$
$$G(x, t) = \frac{1}{4\pi l_{\text{diff}}^2 t} e^{-\frac{x^2}{4l_{\text{diff}}^2 t} - \tau}$$

$l_{\text{diff}} = \sqrt{\tau D}$   
Diffusion length

# Dimensional Crossover: Thermo-Optic Interaction

Photon functional:

$$\mu\psi = \left( -\frac{\hbar^2 \nabla^2}{2m} + V + g_K |\psi|^2 + g_T \int_0^\infty dt \int d^2x' G(\mathbf{x} - \mathbf{x}', t) |\psi(\mathbf{x}')|^2 \right) \psi$$

$g_T = \sigma\gamma\tau B$



→ Energy functional:  $E[\psi^*, \psi] = E_0[\psi^*, \psi] + E_K[\psi^*, \psi] + E_T[\psi^*, \psi]$

- $E_0[\psi^*, \psi] = \int d^2x \left[ \frac{\hbar^2}{2m} |\nabla\psi|^2 + V|\psi|^2 \right]$
- $E_K[\psi^*, \psi] = \frac{g_K}{2} \int d^2x |\psi|^4$
- $E_T[\psi^*, \psi] = \frac{g_T}{2} \int_0^\infty dt \int d^2x \int d^2x' G(\mathbf{x} - \mathbf{x}', t) |\psi(\mathbf{x}')|^2 |\psi(\mathbf{x})|^2$

# Dimensional Crossover: Thermo-Optic Interaction

Variational approach:

$$\lambda = \frac{l_x}{l_y}$$

• Harmonic potential:  $V = \frac{m\Omega^2}{2} (x^2 + \lambda^4 y^2)$

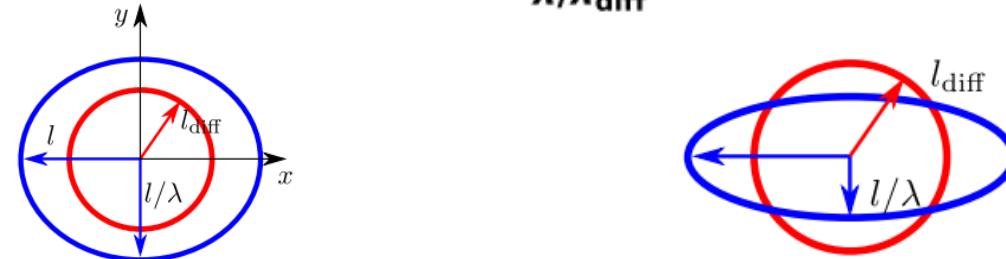
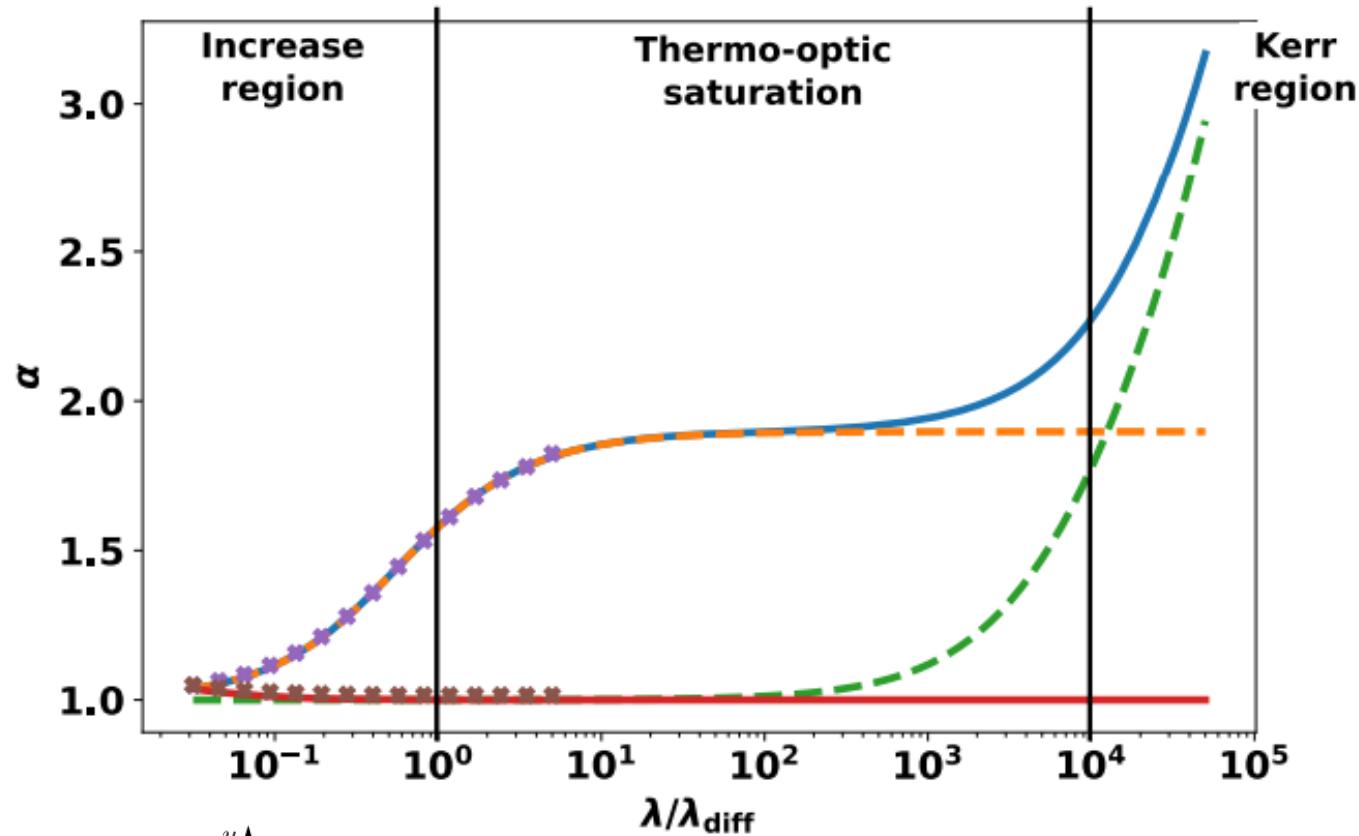
• Gauss ansatz function:

$$\psi = \sqrt{\frac{\lambda N}{\alpha_x \alpha_y \pi l_x^2}} \exp\left(-\frac{1}{2l_x^2} \left(\frac{x^2}{\alpha_x^2} + \lambda^2 \frac{y^2}{\alpha_y^2}\right)\right)$$

→ Algebraic equations for  $\alpha_x, \alpha_y$

# Dimensional Crossover: Thermo-Optic Interaction

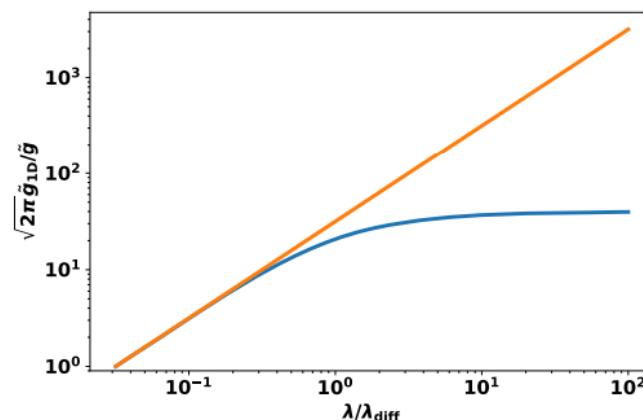
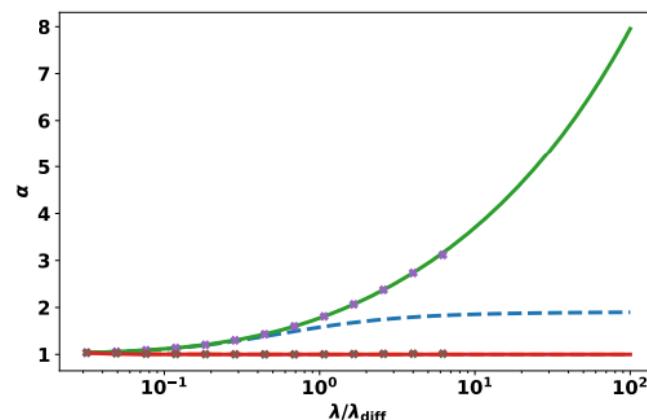
General solution:



# Dimensional Crossover: Thermo-Optic Interaction

Dimensional crossover:

- $\bar{\Omega} = \lambda \Omega_x$
- Harmonic potential:  $V = \frac{m\bar{\Omega}^2}{2} \left( \frac{x^2}{\lambda^2} + \lambda^2 y^2 \right)$
  - Gaussian ansatz function:  $\psi = \sqrt{\frac{\lambda N}{\bar{\alpha}_x \bar{\alpha}_y \pi \bar{l}^2}} \exp \left( -\frac{1}{2\bar{l}^2} \left( \frac{x^2}{\bar{\alpha}_x^2} + \lambda^2 \frac{y^2}{\bar{\alpha}_y^2} \right) \right)$
- Coupled algebraic equations for variational parameters
- $$\bar{l} = \frac{l_x}{\sqrt{\lambda}}$$

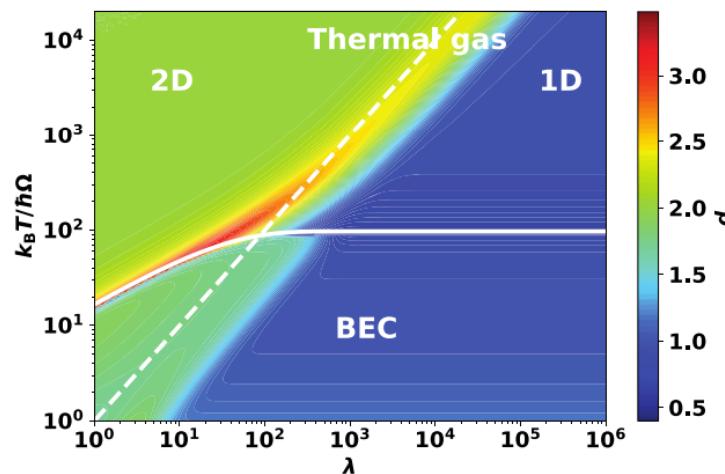
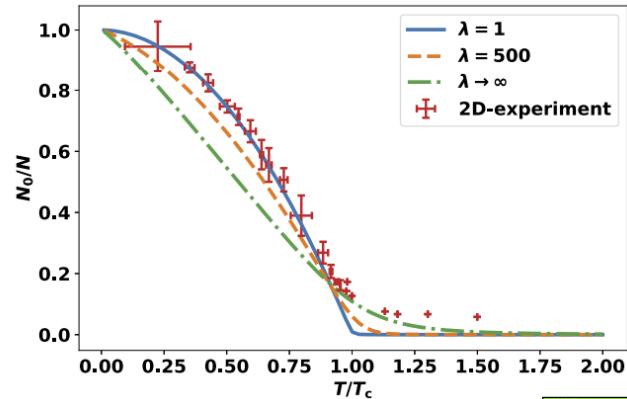


# Summary

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Without interaction:

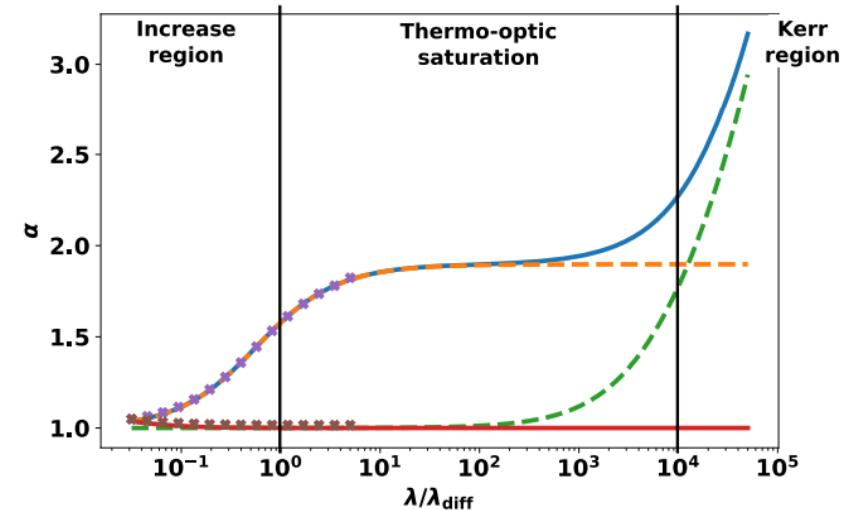
$$\Pi = -\frac{g}{\beta} \sum_{j,n=0}^{\infty} \sum_{k=1}^{\infty} \frac{e^{-\beta(E_{jn}(\lambda) - \mu)k}}{k} = \Pi_{1D} + \Delta\Pi(\lambda)$$



35

With interaction:

$$\mu\psi = \left( -\frac{\hbar^2 \nabla^2}{2m} + V + g_K |\psi|^2 + g_T \int_0^\infty dt \int d^2x' G(x - x', t) |\psi(x')|^2 \right) \psi$$



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