Hydrodynamic Description of Vortices in Photon Bose-Einstein Condensates

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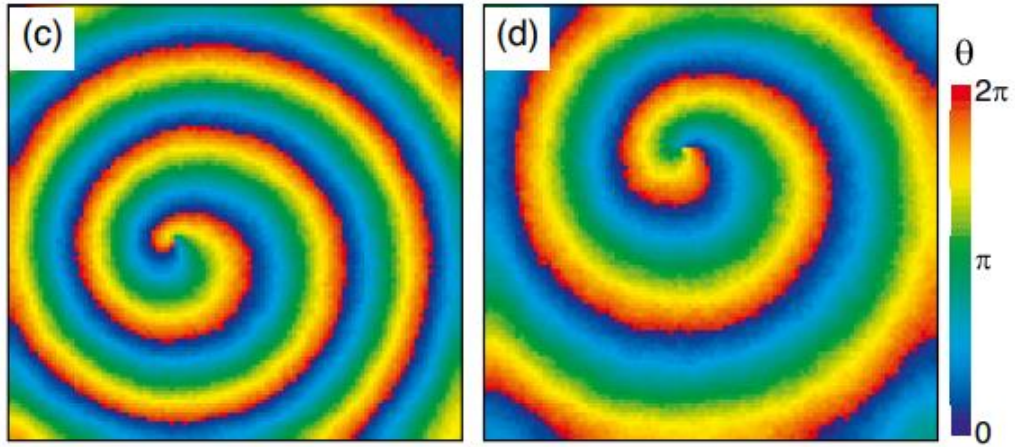
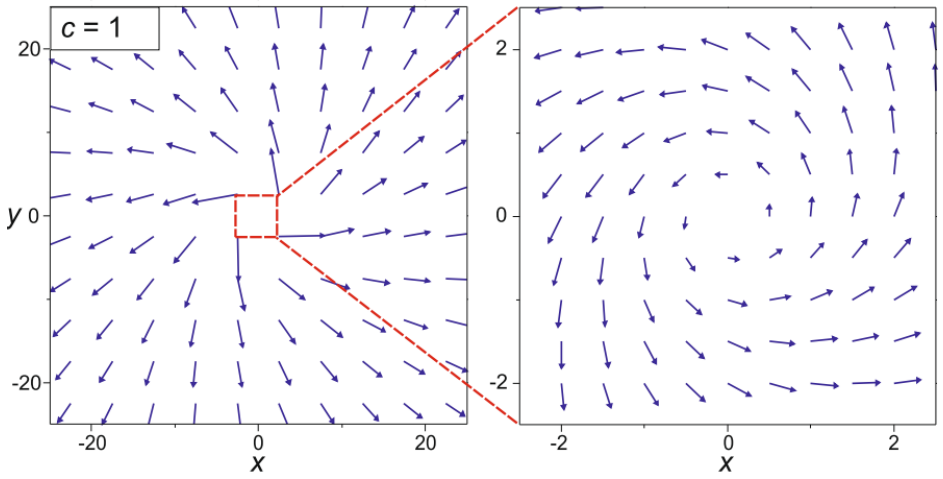
Motivation



M. Wouters, V. N. Gladilin, PRL **125,** 215301 (2020)



M. Wouters, V. N. Gladilin, NJP **19,** 105005 (2017)

How can we analytically describe vortices in photon Bose-Einstein condensates?

Topics

- Hydrodynamic equations
  - For closed systems
  - Extension to open systems
- Variational methods
  - Action based principle
  - Projector method
- Solution without vortex
- Vortex solution
- Summary
- Outlook

Hydrodynamic Equations: Closed System

$$i\hbar \frac{\partial \psi(\boldsymbol{x},t)}{\partial t} = \left\{ -\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2 + U(\boldsymbol{x}) + g|\psi(\boldsymbol{x},t)|^2 \right\} \psi(\boldsymbol{x},t)$$

Madelung transformation: $\psi(\boldsymbol{x},t) = \sqrt{n(\boldsymbol{x},t)}e^{i\Phi(\boldsymbol{x},t)}$

$$\frac{\partial n(\boldsymbol{x},t)}{\partial t} = -\frac{\hbar}{m}\left[\boldsymbol{\nabla} n(\boldsymbol{x},t)\boldsymbol{\nabla}\Phi(\boldsymbol{x},t) + n(\boldsymbol{x},t)\boldsymbol{\nabla}^2\Phi(\mathrm{x},\mathrm{t})\right]$$

$$\frac{\partial \Phi(\boldsymbol{x},t)}{\partial t} = \frac{\hbar}{2m}\left\{ \frac{\boldsymbol{\nabla}^2\sqrt{n(\boldsymbol{x},t)}}{\sqrt{n(\boldsymbol{x},t)}} - [\boldsymbol{\nabla}\Phi(\boldsymbol{x},t)]^2 \right\} - \frac{U(\boldsymbol{x})}{\hbar} - \frac{gn(\boldsymbol{x},t)}{\hbar}$$

Hydrodynamic Equations: Closed System

Introduce velocity field: $\boldsymbol{v}(\boldsymbol{x},t) = \frac{\hbar}{m}\boldsymbol{\nabla}\Phi(\boldsymbol{x},t)$

Continuity equation

$$\frac{\partial n(\boldsymbol{x},t)}{\partial t} + \boldsymbol{\nabla}\cdot[\underbrace{n(\boldsymbol{x},t)\boldsymbol{v}(\boldsymbol{x},t)}_{\text{Current density}}] = 0$$

$$m\frac{\partial \boldsymbol{v}(\boldsymbol{x},t)}{\partial t} = -\boldsymbol{\nabla}\left\{\frac{1}{2}m\boldsymbol{v}(\boldsymbol{x},t)^2 + U(\boldsymbol{x}) + gn(\boldsymbol{x},t) - \frac{\hbar^2}{2m}\frac{\boldsymbol{\nabla}^2\sqrt{n(\boldsymbol{x},t)}}{\sqrt{n(\boldsymbol{x},t)}}\right\}$$

Newton equation

$$\boldsymbol{\nabla}\left(\frac{\boldsymbol{v}\cdot\boldsymbol{v}}{2}\right) = (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} - (\boldsymbol{\nabla}\times\boldsymbol{v})\times\boldsymbol{v}$$

$$\frac{\partial \boldsymbol{v}(\boldsymbol{x},t)}{\partial t} + (\boldsymbol{v}(\boldsymbol{x},t)\cdot\boldsymbol{\nabla})\boldsymbol{v}(\boldsymbol{x},t) = -\boldsymbol{\nabla}\left[\frac{U(\boldsymbol{x})}{m} + \frac{gn(\boldsymbol{x},t)}{m} - \frac{\hbar^2}{2m^2}\frac{\boldsymbol{\nabla}^2\sqrt{n(\boldsymbol{x},t)}}{\sqrt{n(\boldsymbol{x},t)}}\right] + \left(\boldsymbol{\nabla}\times\boldsymbol{v}(\boldsymbol{x},t)\right)\times\boldsymbol{v}(\boldsymbol{x},t)$$

Euler equation/ Quantum Bernoulli equation

Vorticity: $\boldsymbol{\omega}(\boldsymbol{x},t) = \boldsymbol{\nabla}\times\boldsymbol{v}(\boldsymbol{x},t)$

$$\frac{\partial \boldsymbol{\omega}(\boldsymbol{x},t)}{\partial t} + (\boldsymbol{v}(\boldsymbol{x},t)\cdot\boldsymbol{\nabla})\boldsymbol{\omega}(\boldsymbol{x},t) = (\boldsymbol{\omega}(\boldsymbol{x},t)\cdot\boldsymbol{\nabla})\boldsymbol{v}(\boldsymbol{x},t) - \boldsymbol{\omega}(\boldsymbol{x},t)\cdot\left(\boldsymbol{\nabla}\cdot\boldsymbol{v}(\boldsymbol{x},t)\right)$$

Helmholtz vorticity equation

Vortex in 2D: Closed System

Phase for $\boldsymbol{x} \neq \boldsymbol{0}$: $\varphi_S(\boldsymbol{x}) = \arctan\left(\frac{y}{x}\right)$ ➡ Velocity for $\boldsymbol{x} \neq \boldsymbol{0}$: $\boldsymbol{v}(\boldsymbol{x}) = \frac{\hbar}{m}\boldsymbol{\nabla}\varphi_S(\boldsymbol{x}) = \frac{\hbar}{m}\frac{\boldsymbol{e}_\varphi}{r}$

Singularity at origin:

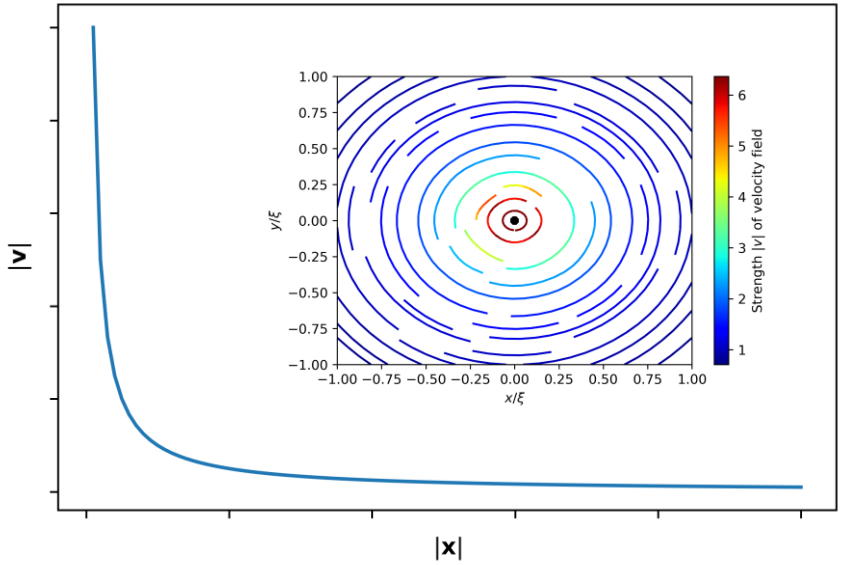
$$\boldsymbol{\nabla} \times \boldsymbol{v} = \left(\frac{\partial^2 \varphi_S}{\partial x \partial y} - \frac{\partial^2 \varphi_S}{\partial y \partial x}\right)\boldsymbol{e}_z \neq \boldsymbol{0}$$

➡ Theorem of Schwarz not valid.

Circulation:

$$\int_A (\boldsymbol{\nabla} \times \boldsymbol{v}) \cdot d\boldsymbol{A} = \oint_C \boldsymbol{v} \cdot d\boldsymbol{x} = \frac{\hbar}{m}\oint_C \boldsymbol{\nabla}\varphi_S \cdot d\boldsymbol{x} = 2\pi\frac{\hbar}{m}$$

➡ $$\boldsymbol{\nabla} \times \boldsymbol{v} = 2\pi\frac{\hbar}{m}\delta(\boldsymbol{x})\boldsymbol{e}_z$$

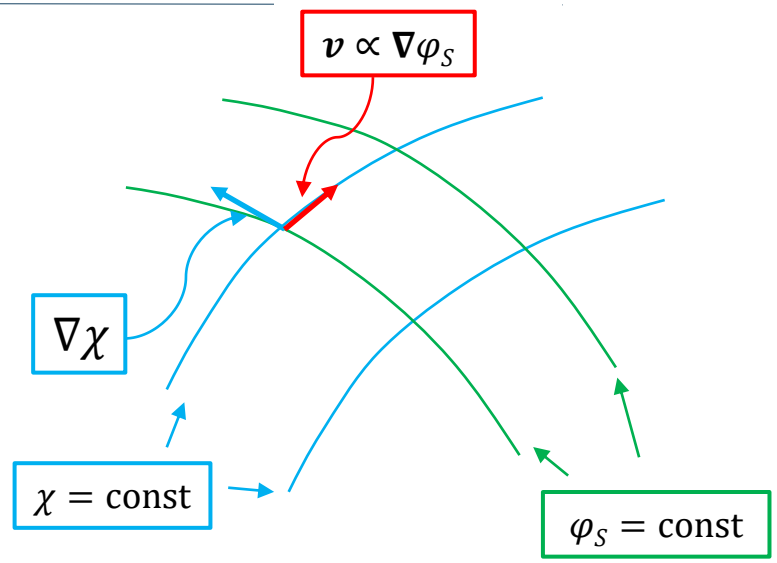


Vortex in 2D: Stream Function

Introduce function $\chi(\boldsymbol{x})$:

$$\boldsymbol{v}(\boldsymbol{x}) = \frac{\hbar}{m}\nabla\varphi_S(\boldsymbol{x}) = \nabla \times \left[-\frac{\hbar}{m}\chi(\boldsymbol{x})\boldsymbol{e}_z\right]$$

singular regular



$$\nabla \times \boldsymbol{v}(\boldsymbol{x}) = \frac{\hbar}{m}\Delta\chi(\boldsymbol{x})\boldsymbol{e}_z = 2\pi\frac{\hbar}{m}\delta(\boldsymbol{x})\boldsymbol{e}_z$$

$$\boxed{\Delta\chi(\boldsymbol{x}) = 2\pi\delta(\boldsymbol{x})}$$

Poisson equation

$$\boxed{\chi(\boldsymbol{x}) = \frac{1}{2\pi}\ln(|\boldsymbol{x}|) + C}$$

Green's function in 2D

Yield same velocity: $\boldsymbol{v}(\boldsymbol{x}) = \nabla \times \left[-\frac{\hbar}{m}\chi(\boldsymbol{x})\boldsymbol{e}_z\right] = \frac{\hbar}{m}\frac{\boldsymbol{e}_\varphi}{r}$

Vortex in 2D: Open System

$$\boldsymbol{v}(\boldsymbol{x}) = \frac{\hbar}{m}\boldsymbol{\nabla}\varphi_S(\boldsymbol{x}) + \frac{\hbar}{m}\boldsymbol{\nabla}\varphi_R(\boldsymbol{x})$$

$$\frac{\partial^2\varphi_S}{\partial x\partial y} - \frac{\partial^2\varphi_S}{\partial y\partial x} \neq 0 \longleftarrow \text{singular} \qquad \text{regular} \longrightarrow \frac{\partial^2\varphi_R}{\partial x\partial y} - \frac{\partial^2\varphi_R}{\partial y\partial x} = 0$$

$$\Rightarrow \quad \boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{\nabla} \times \left[-\frac{\hbar}{m}\chi(\boldsymbol{x})\boldsymbol{e}_z\right] + \frac{\hbar}{m}\boldsymbol{\nabla}\varphi_R(\boldsymbol{x})$$

Helmholtz vector decomposition theorem in 2D

H. Helmholtz, *Journal für die reine und angewandte Mathematik*, 1858

Hydrodynamic Equations: Open System

$$i\hbar\frac{\partial\psi(\boldsymbol{x},t)}{\partial t}=\left\{-\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2+U(\boldsymbol{x})+g|\psi(\boldsymbol{x},t)|^2+i[\gamma-\Gamma|\psi(\boldsymbol{x},t)|^2]\right\}\psi(\boldsymbol{x},t)$$

Gross-Pitaevski equation

Particle rate of gain

Density dependent rate of loss

J. Keeling, N. G. Berloff, PRL **100**, 2008

$$\frac{\partial n(\boldsymbol{x},t)}{\partial t}+\boldsymbol{\nabla}[n(\boldsymbol{x},t)\boldsymbol{v}(\boldsymbol{x},t)]=2n(\boldsymbol{x},t)[\gamma-\Gamma n(\boldsymbol{x},t)]$$

Continuity equation

Integration yields: $\frac{\partial}{\partial t}\int d^3x\, n(\boldsymbol{x},t)=2\int d^3x\{n(\boldsymbol{x},t)\gamma-\Gamma n(\boldsymbol{x},t)^2\}$

$$0=\int d^3x\{n(\boldsymbol{x})\gamma-\Gamma n(\boldsymbol{x})^2\}$$

Steady state continuity equation

RPTU

Hamilton Variational Principle

Action: $\mathcal{A} = \mathcal{A}[\psi, \psi^*] = \int dt \int d^n x \, \mathcal{L}[\psi, \psi^*, \partial\psi, \partial\psi^*, \boldsymbol{x}, t]$

$$\Rightarrow \quad \frac{\delta \mathcal{A}}{\delta \psi^*(\boldsymbol{x}, t)} = M[\psi, \psi^*](\boldsymbol{x}, t) = 0$$

Variational approach: $\psi(\boldsymbol{x}, t) \approx \Psi(\boldsymbol{x}, \{\lambda_i(t)\})$

$$\Rightarrow \quad \mathcal{A} \approx \mathcal{A}[\Psi(\{\lambda_i(t)\}), \Psi^*(\{\lambda_i(t)\})] = \mathcal{A}(\{\lambda_i(t)\})$$

$$\Rightarrow \quad \frac{\delta \mathcal{A}}{\delta \lambda_i(t)} = 0 \qquad \forall i$$

$\Rightarrow$ Equation for variational parameters

V. M. Perez-Garcia, H. Michinel, J. I. Cirac, M. Lewenstein, P. Zoller, PRL **77**, 1996

Not applicable for open systems, because no action exists.

Projector Method

Approximate wave functions: $\psi(\boldsymbol{x},t) \approx \Psi(\boldsymbol{x},\{\lambda_i(t)\})$

Assume action exists:

Chain rule

Known equation of motion

$$\frac{\delta\mathcal{A}[\Psi(\{\lambda_i(t)\}),\Psi^*(\{\lambda_i(t)\})]}{\delta\lambda_i(t)} = \int d^n x \left[\frac{\delta\mathcal{A}}{\delta\Psi(\boldsymbol{x},\{\lambda_i(t)\})}\cdot\frac{\partial\Psi(\boldsymbol{x},\{\lambda_i(t)\})}{\partial\lambda_i(t)} + \frac{\delta\mathcal{A}}{\delta\Psi^*(\boldsymbol{x},\{\lambda_i(t)\})}\cdot\frac{\partial\Psi^*(\boldsymbol{x},\{\lambda_i(t)\})}{\partial\lambda_i(t)}\right] = 0$$

Known from ansatz function

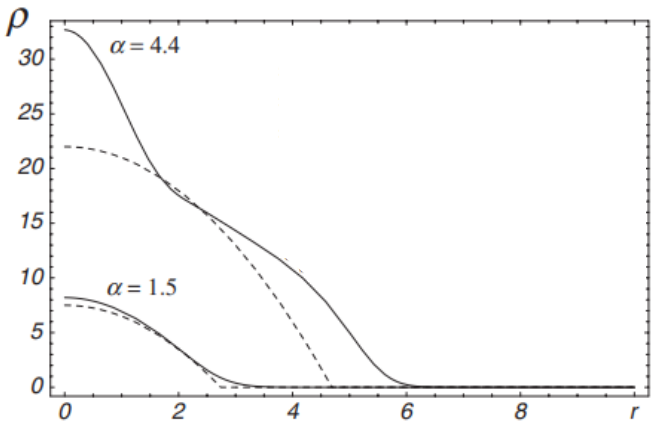
$$\boxed{0 = \int d^n x \left[M^*[\Psi,\Psi^*](\boldsymbol{x},t)\cdot\frac{\partial\Psi(\boldsymbol{x},\{\lambda_i(t)\})}{\partial\lambda_i(t)} + M[\Psi,\Psi^*](\boldsymbol{x},t)\cdot\frac{\partial\Psi^*(\boldsymbol{x},\{\lambda_i(t)\})}{\partial\lambda_i(t)}\right]}$$

Projector method

M. dos Santos Filho, F. E. A. dos Santos, private communication

- Equation of motion
- Ansatz for wave functions **without restriction**
- Calculation of projector integrals

Projector Method vs. Cumulant Method



J. Keeling, N. G. Berloff, PRL **100**, 2008

$$i\hbar \frac{\partial \psi(\boldsymbol{x},t)}{\partial t} = \left\{ -\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2 + \frac{1}{2}m\omega^2\boldsymbol{x}^2 + g|\psi(\boldsymbol{x},t)|^2 + i[\gamma - \Gamma|\psi(\boldsymbol{x},t)|^2] \right\} \psi(\boldsymbol{x},t)$$

Variational steady state ansatz function for small pumping:

$$\psi(\boldsymbol{r},t) = \sqrt{\frac{N}{\pi q^2}}\; e^{-\frac{r^2}{2q^2}}\; e^{-\frac{i}{\hbar}\mu t}$$

Cumulant method:

$$\cdot \int d^3x\, \psi^* \longrightarrow N = \ldots \qquad \mu = \ldots$$

$$\cdot \int d^3x\, r^2 \psi^* \longrightarrow q^2 = \ldots$$

Dimensionless quantities:

$$\alpha = \frac{2\gamma}{\hbar\omega}\;, \sigma = \frac{\Gamma}{g}$$

E. Stein, F. Vewinger, A. Pelster, NPJ **10**, 2019

N. Mann, M. R. Bakhtiari, A. Pelster, M. Thorwart, PRL **120**, 2018

Projector Method vs. Cumulant Method

Calculating N, μ, q^2 for both methods yield the same equations.

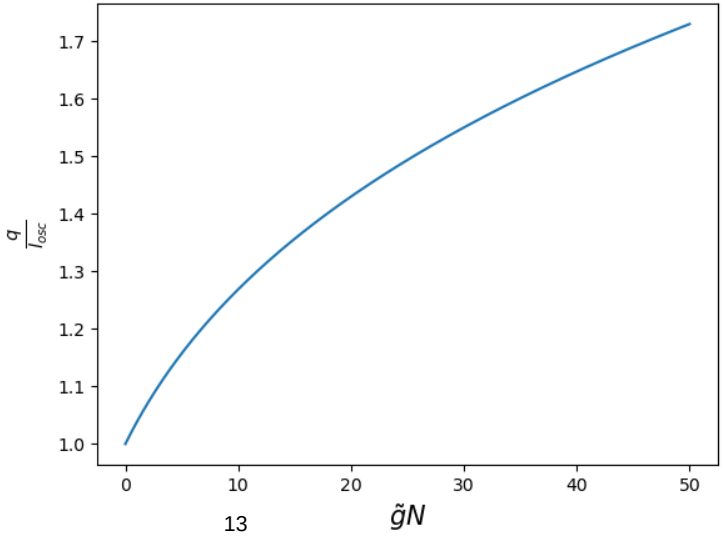
$$q = \sqrt[4]{\frac{\tilde{g}N}{2\pi} + 1} \cdot l_{osc}$$

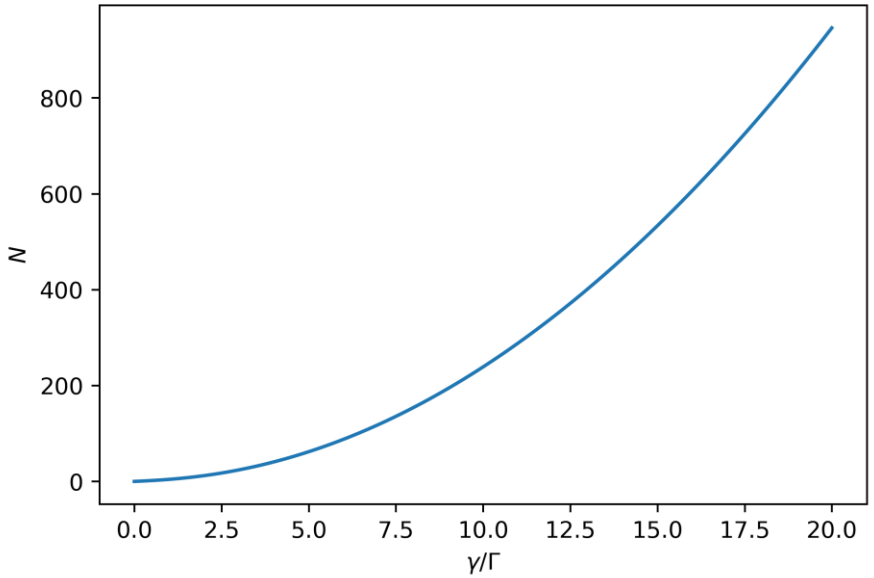
$$N = \frac{2\pi\gamma q^2}{\Gamma}$$

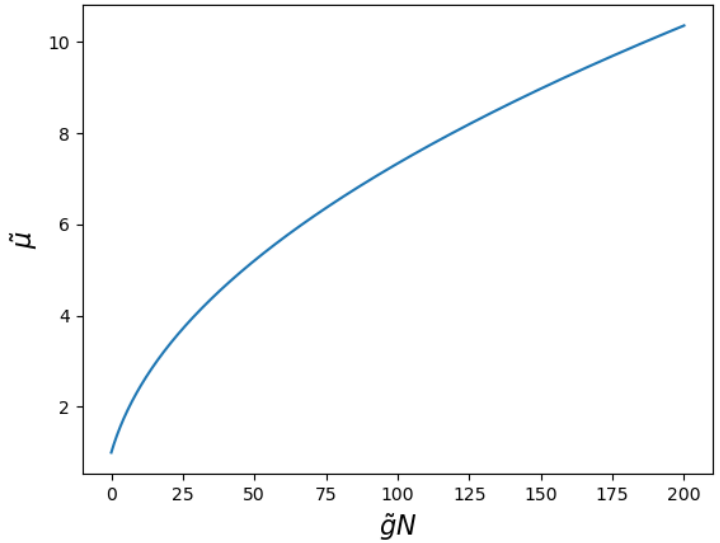
$$\tilde{\mu} = \frac{1}{2\sqrt{\frac{\tilde{g}N}{2\pi} + 1}} + \frac{1}{2}\sqrt{\frac{\tilde{g}N}{2\pi} + 1} + \frac{\tilde{g}N}{2\pi\sqrt{\frac{\tilde{g}N}{2\pi} + 1}}$$

$$\tilde{\mu} = \frac{\mu}{\hbar\omega}$$

$$\tilde{g} = \frac{gm}{\hbar^2}, \qquad l_{osc} = \sqrt{\frac{\hbar}{m\omega}}$$







Vortex Solution

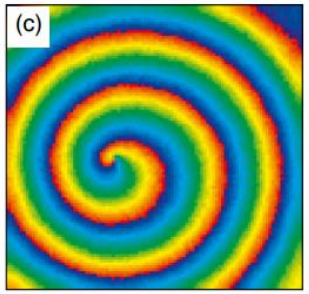
$$i\hbar\frac{\partial\psi(\boldsymbol{x},t)}{\partial t}=\left\{-\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2+g|\psi(\boldsymbol{x},t)|^2+i[\gamma-\Gamma|\psi(\boldsymbol{x},t)|^2]\right\}\psi(\boldsymbol{x},t)$$

$$\psi(r,\varphi,t)=\psi(r,\varphi)e^{-\frac{i}{\hbar}\mu t}$$

Steady states

$$\psi(r,\varphi)=\sqrt{n(r)}e^{i\Phi(r,\varphi)}$$

Madelung transformation



M. Wouters, V. N. Gladilin,
PRL **125,** 215301 (2020)

Incompressible part

Singular phase

$$\boldsymbol{v}(r,\varphi)=\frac{\hbar}{m}\boldsymbol{\nabla}\Phi(r,\varphi)=\frac{\hbar}{m}\boldsymbol{\nabla}\varphi_S(\varphi)+\frac{\hbar}{m}\boldsymbol{\nabla}\varphi_R(r)$$

Compressible part

Regular phase

Helmholtz decomposition theorem in 2D

Vortex Solution

Large distance behaviour yields relation between μ, g, γ, Γ :

$$\boxed{\frac{\mu}{g} = \frac{\gamma}{\Gamma} = n_\infty}$$

Introducing dimensionless quantities:

$$g n_\infty = \frac{\hbar^2}{2m\xi^2}, \quad \chi(r) = \sqrt{\frac{n(r)}{n_\infty}}, \quad r \to \frac{r}{\xi}$$

Yields dimensionless equation of motion:

$$M[\chi, \varphi_R](r) = \chi(r) e^{i\Phi(r,\varphi)} + \left\{ \left[\nabla^2 \chi(r) - \frac{\chi(r)}{r^2} - \chi(r)\left(\nabla\varphi_R(r)\right)^2 \right] + i\left[\chi(r)\nabla^2\varphi_R(r) + 2\nabla\chi(r)\nabla\varphi_R(r)\right] \right\} e^{i\Phi(r,\varphi)}$$
$$-\chi^3(r) e^{i\Phi(r,\varphi)} - i\frac{\Gamma}{g}\left[1 - \chi^2(r)\right]\chi(r) e^{i\Phi(r,\varphi)}$$

One dimensionless parameter in system

Vortex Solution

Consider variational ansatz function: $\chi(r) = \dfrac{r}{\sqrt{r^2+\alpha}}$

Treat $\varphi_R(r)$ as variational function:

Projector method yields:

$$\boxed{\frac{1}{4} - \frac{1}{2\alpha} = \int_{-\infty}^{+\infty} dr \frac{r^3}{(r^2+\alpha)^2}\left[\frac{d\varphi_R(r)}{dr}\right]^2}$$

$$\boxed{\frac{d^2\varphi_R(r)}{dr^2} + \frac{r^2+3\alpha}{r(r^2+\alpha)}\frac{d\varphi_R(r)}{dr} = \frac{\Gamma}{g}\frac{\alpha}{r^2+\alpha}}$$

$$\frac{d\varphi_R(r)}{dr} = C\left[\frac{r^2+\alpha}{r^3}\right] + \left[\frac{\alpha\Gamma}{2g}\frac{\alpha}{r^3} + \frac{\alpha\Gamma}{2g}\ln(r^2+\alpha)\frac{r^2+\alpha}{r^3}\right],$$

Avoid divergencies: $C = -\dfrac{\alpha\Gamma}{2g}[1+\ln(\alpha)]$

$$\Rightarrow \boxed{\frac{d\varphi_R(r)}{dr} = \frac{\alpha\Gamma}{2g}\left[\ln\left(\frac{r^2+\alpha}{\alpha}\right)\frac{r^2+\alpha}{r^3} - \frac{1}{r}\right]}$$

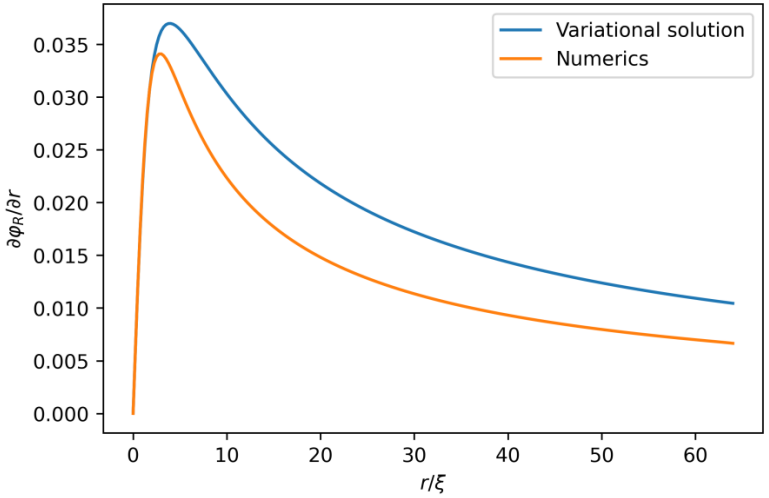
$$\Rightarrow \boxed{\alpha = \left(\frac{g}{\Gamma}\right)^2\left[1 - \sqrt{1 - 4\left(\frac{\Gamma}{g}\right)^2}\right]}$$

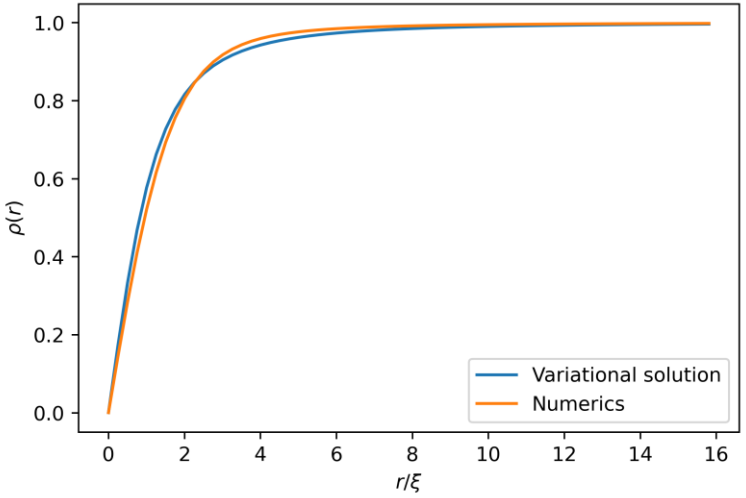
$\Rightarrow$ Restriction: $0 \leq \left|\dfrac{\Gamma}{g}\right| \leq \dfrac{1}{2}$

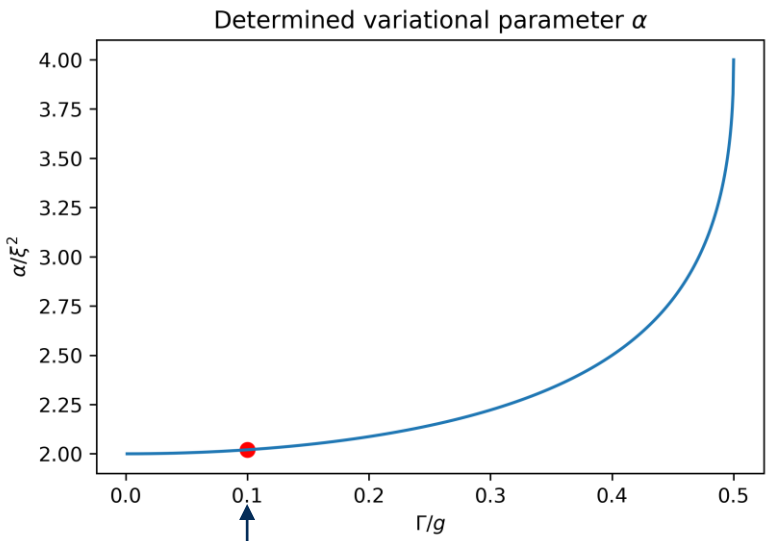
Vortex Solution

$$\frac{d\varphi_R(r)}{dr} = \frac{\alpha\Gamma}{2g}\left[\ln\left(\frac{r^2+\alpha}{\alpha}\right)\frac{r^2+\alpha}{r^3} - \frac{1}{r}\right]$$

$$\alpha = \left(\frac{g}{\Gamma}\right)^2\left[1 - \sqrt{1 - 4\left(\frac{\Gamma}{g}\right)^2}\right]$$



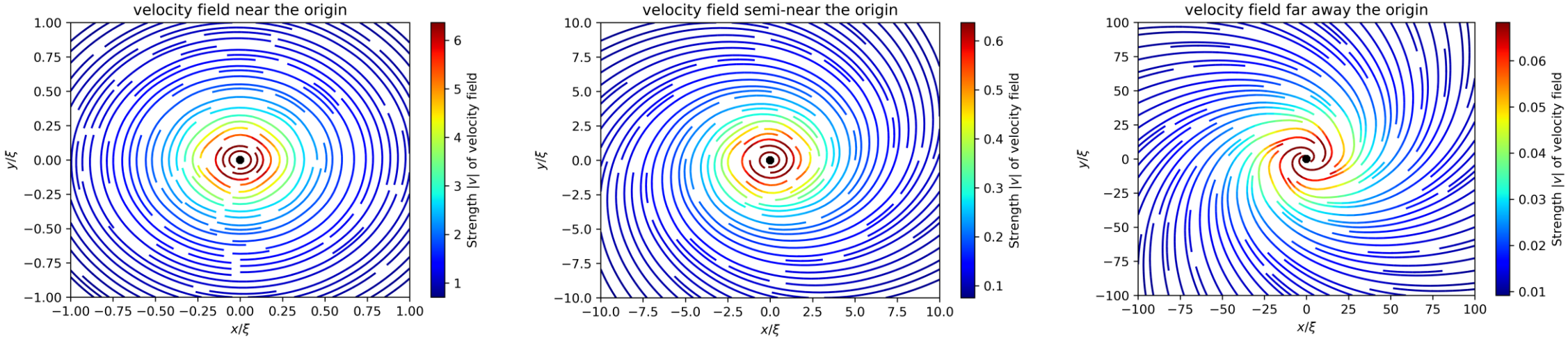




Used $\frac{\Gamma}{g}$

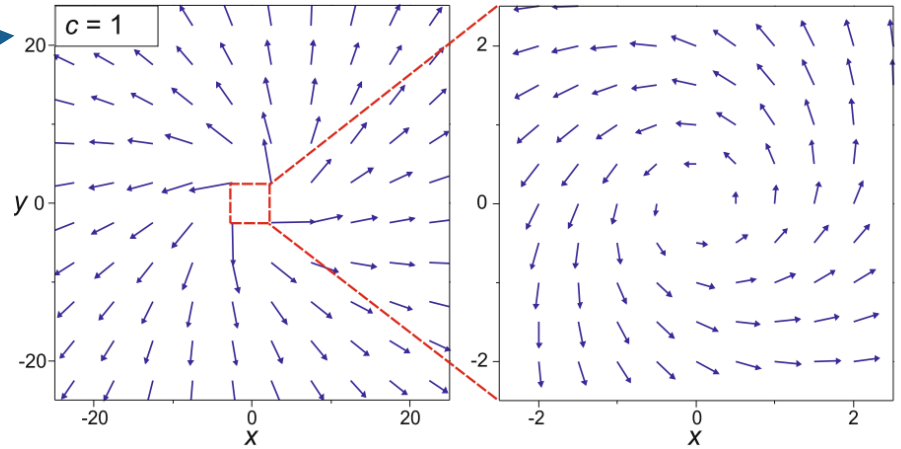
Help for numerics: M. dos Santos Filho

Vortex Solution



Compare determined velocity with M. Wouters, V. N. Gladilin, NJP **19,** 105005 (2017):

$$c \propto \frac{\Gamma}{g}$$



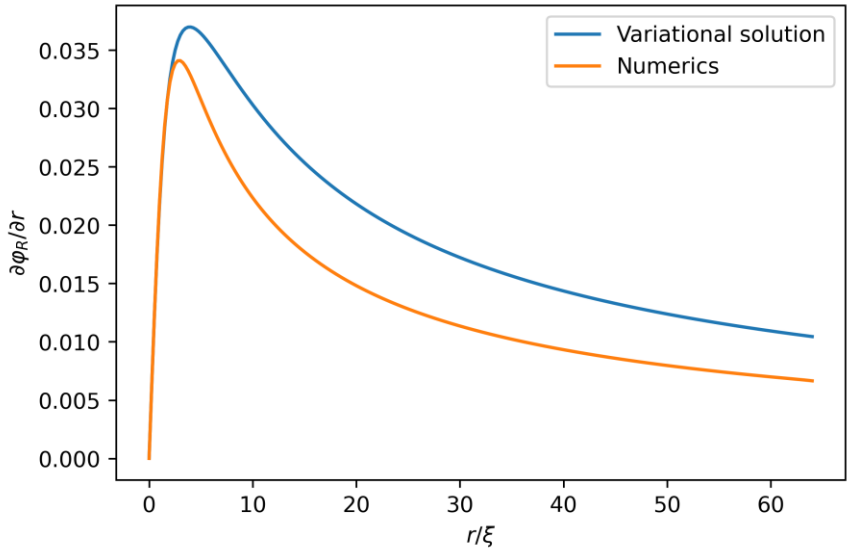
Summary

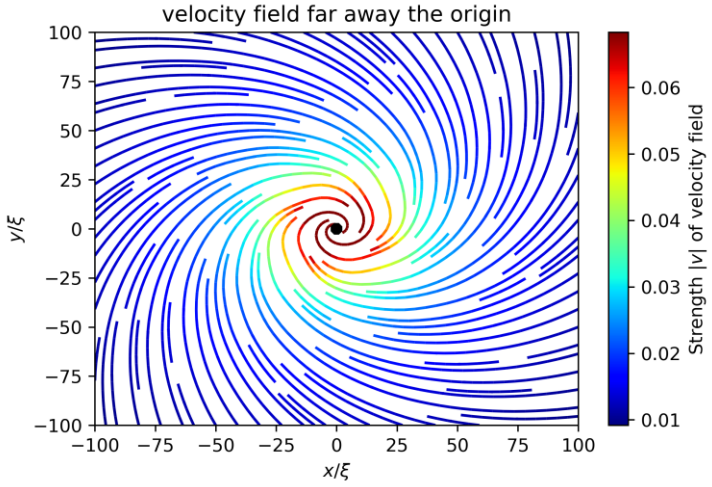
Extended action based variational method to open systems.

$$0 = \int d^n x \left[M\left[\Psi, \Psi^*\right](\boldsymbol{x}, t) \cdot \frac{\partial \Psi^*(\boldsymbol{x}, \{\lambda_i(t)\})}{\partial \lambda_i(t)} + M^*\left[\Psi, \Psi^*\right](\boldsymbol{x}, t) \cdot \frac{\partial \Psi\,(\boldsymbol{x}, \{\lambda_i(t)\})}{\partial \lambda_i(t)} \right]$$

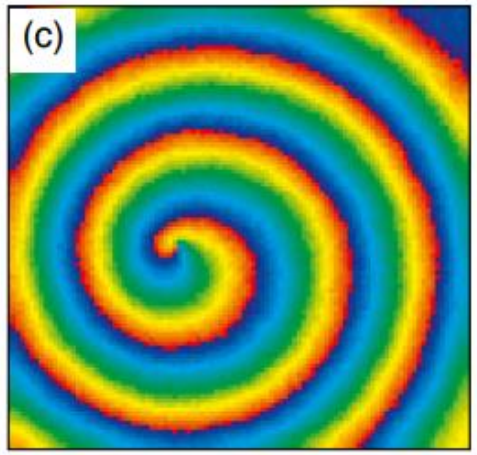
Found vortex solution by using Helmholtz decomposition.

$$\boldsymbol{v}(r, \varphi) = \frac{\hbar}{m} \boldsymbol{\nabla} \varphi_S(\varphi) + \frac{\hbar}{m} \boldsymbol{\nabla} \varphi_R(r)$$





Outlook & Preliminary Results



M. Wouters, V. N. Gladilin, PRL **125,** 215301 (2020)

Comparison of Wouters model with Keeling & Berloff model:

$$i\hbar\frac{\partial\psi(\boldsymbol{x},t)}{\partial t} = -(1-i\kappa)J\sum_{\boldsymbol{x}'\in\mathcal{N}_x}\psi(\boldsymbol{x}',t) + V(\boldsymbol{x})\psi(\boldsymbol{x},t) + \frac{i}{2}[B_{21}M_2(\boldsymbol{x}) - B_{12}M_1(\boldsymbol{x}) - \gamma]\psi(\boldsymbol{x},t)$$

$$\kappa = \frac{1}{2k_BT}B_{12}M_1 = 0.763$$

$$\frac{B_{12}}{B_{21}} = e^{\frac{\hbar\omega}{k_BT}}$$

$$i\hbar\frac{\partial\psi(\boldsymbol{x},t)}{\partial t} = \left[-\frac{\hbar^2\nabla^2}{2m} + V(\boldsymbol{x}) + g|\psi(\boldsymbol{x},t)|^2 + i(\gamma - \Gamma|\psi(\boldsymbol{x},t)|^2)\right]\psi(\boldsymbol{x},t)$$

Consider therefore continuous Wouters model:

$$\boxed{i\hbar\frac{\partial\psi(\boldsymbol{x},t)}{\partial t} = \left[-(1-i\kappa)\frac{\hbar^2\nabla^2}{2m} + \frac{1}{2}m\omega^2|\boldsymbol{x}|^2 + g|\psi(\boldsymbol{x},t)|^2 + i(\gamma - \Gamma|\psi(\boldsymbol{x},t)|^2)\right]\psi(\boldsymbol{x},t)}$$

Ansatz: no vortex

$$\psi(\boldsymbol{r},t) = \sqrt{\frac{N}{\pi q^2}}\; e^{-\frac{r^2}{2q^2} - iAr^2}\; e^{-\frac{i}{\hbar}\mu t}$$

$$\frac{q}{l_{osc}} = \sqrt{\frac{2\kappa A l_{osc}^2}{4A^2 l_{osc}^4 + 1} + \sqrt{\left(\frac{2\kappa A l_{osc}^2}{4A^2 l_{osc}^4 + 1}\right)^2 + \frac{\frac{\tilde{g}N}{2\pi} + 1}{4A^2 l_{osc}^4 + 1}}}$$