

# Spinor Bose Gases in Cubic Optical Lattice

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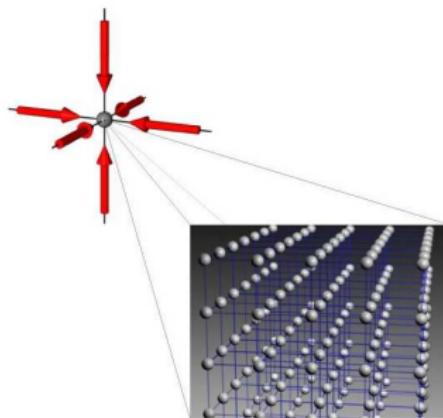
Fachbereich Physik

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# Outline of the Talk

- 1 Experimental Realization
- 2 Bose-Hubbard Model for Spin-1 Boson
- 3 Ginzburg-Landau Theory
- 4 Quantum Phase Boundary
- 5 Magnetic Superfluid Phases
- 6 Order of Phase Transitions
- 7 Summary and Outlook

# Experimental Realization



- Counter-propagating laser beams produce periodic potential, e.g.

$$V(\mathbf{x}) = V_0 \sum_{j=1}^D \sin^2(2\pi x_j / \lambda)$$

- Dimensionality  $\Rightarrow$  tunable
- Lattice geometry and potential depth  $\Rightarrow$  controllable

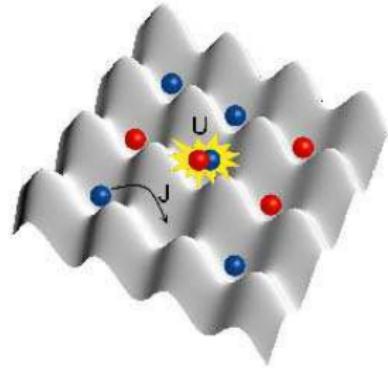
# Bose-Hubbard Model for Spin-1 Boson

**BH Hamiltonian:**

$$\hat{H}_{\text{BH}} = \hat{H}^{(0)} + \hat{H}^{(1)}$$

$$\hat{H}^{(0)} = \sum_i \left[ \frac{U_0}{2} \hat{n}_i (\hat{n}_i - 1) + \frac{U_2}{2} (\hat{\mathbf{S}}_i^2 - 2\hat{n}_i) - \mu \hat{n}_i - \eta \hat{S}_{iz} \right]$$

$$\hat{H}^{(1)} = - \sum_{ij} J_{ij} \sum_{\alpha} \hat{a}_{i\alpha}^\dagger \hat{a}_{j\alpha}$$



- $\hat{n}_i = \sum_{\alpha} \hat{n}_{i\alpha} = \sum_{\alpha} \hat{a}_{i\alpha}^\dagger \hat{a}_{i\alpha}$

- $\left[ \hat{a}_{i\alpha}, \hat{a}_{j\beta} \right] = \left[ \hat{a}_{i\alpha}^\dagger, \hat{a}_{j\beta}^\dagger \right] = 0$

$$\left[ \hat{a}_{i\alpha}, \hat{a}_{j\beta}^\dagger \right] = \delta_{\alpha,\beta} \delta_{i,j}$$

- $\hat{\mathbf{S}}_i = \sum_{\alpha,\beta} \hat{a}_{i\alpha}^\dagger \mathbf{F}_{\alpha\beta} \hat{a}_{i\beta}$

- $U_2 = \begin{cases} > 0, & \text{Anti-ferromagnetic, e.g. } {}^{23}\text{Na} \\ < 0, & \text{Ferromagnetic, e.g. } {}^{87}\text{Rb} \end{cases}$

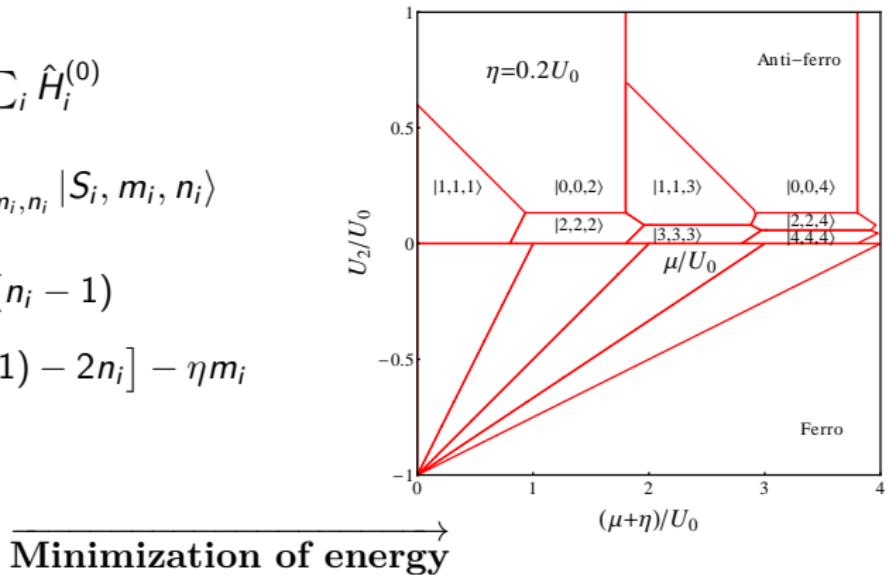
- $J_{ij} = \begin{cases} J, & \text{if } i, j \text{ are next neighbors} \\ 0, & \text{otherwise} \end{cases}$

# Atomic Limit ( $J = 0$ ): Mott Phases

- Site-diagonal:  $\hat{H}^{(0)} = \sum_i \hat{H}_i^{(0)}$

$$\hat{H}_i^{(0)} |S_i, m_i, n_i\rangle = E_{S_i, m_i, n_i}^{(0)} |S_i, m_i, n_i\rangle$$

$$E_{S_i, m_i, n_i}^{(0)} = -\mu n_i + \frac{U_0}{2} n_i(n_i - 1) + \frac{U_2}{2} [S_i(S_i + 1) - 2n_i] - \eta m_i$$



1.  $\eta > 0 \Rightarrow m_i = S_i$

2. Two competing effects:

$\eta > 0 \hat{=} \text{maximize } S_i$       ↲ ↳       $U_2 > 0 \hat{=} \text{minimize } S_i$

# Symmetry-Breaking Source

- Hamiltonian with inhomogeneous sources:

$$\hat{H}(\tau) [j, j^*] = \hat{H}_{\text{BH}} + \sum_i \sum_{\alpha} \left[ j_{i\alpha}^*(\tau) \hat{a}_{i\alpha} + j_{i\alpha}(\tau) \hat{a}_{i\alpha}^\dagger \right]$$

- Decomposition in view of perturbative treatment:

$$\hat{H}(\tau) [j, j^*] = \underbrace{\hat{H}^{(0)}_{\text{unperturbed}}}_{\text{unperturbed}} - \underbrace{\sum_{ij} J_{ij} \sum_{\alpha} \hat{a}_{i\alpha}^\dagger \hat{a}_{j\alpha}}_{\text{perturbed}} + \underbrace{\sum_i \sum_{\alpha} \left[ j_{i\alpha}^*(\tau) \hat{a}_{i\alpha} + j_{i\alpha}(\tau) \hat{a}_{i\alpha}^\dagger \right]}_{\text{perturbed}}$$

## Matrix elements

$$\hat{a}_{\alpha}^\dagger |S_i, m_i, n_i\rangle = M_{\alpha, S_i, m_i, n_i} |S_i + 1, m_i + \alpha, n_i + 1\rangle + N_{\alpha, S_i, m_i, n_i} |S_i - 1, m_i + \alpha, n_i + 1\rangle$$

$$\hat{a}_{\alpha} |S_i, m_i, n_i\rangle = O_{\alpha, S_i, m_i, n_i} |S_i + 1, m_i - \alpha, n_i - 1\rangle + P_{\alpha, S_i, m_i, n_i} |S_i - 1, m_i - \alpha, n_i - 1\rangle$$

S. Tsuchiya et. al, Phys. Rev. A **70**, 043628 (2004)

M. Ohliger, Diploma thesis, FUB (2008)

M. Mobarak (FU Berlin)

January 14, 2014

# Generating Functional

Partition function in Dirac Interaction Picture ( $\hbar = 1$ ):

$$\mathcal{Z}[j, j^*] = \text{Tr} \left\{ \hat{T} e^{- \int_0^\beta d\tau \hat{H}(\tau)} [j, j^*] \right\} = \text{Tr} \left\{ e^{-\beta \hat{H}^{(0)}} \hat{U}_D(\beta, 0) \right\}$$

Full partition function

$$\mathcal{Z}[j, j^*] = \mathcal{Z}^{(0)} \left\langle \hat{U}_D(\beta, 0) \right\rangle^{(0)}, \quad \langle \bullet \rangle^{(0)} = \frac{1}{\mathcal{Z}^{(0)}} \text{Tr} \left[ \bullet \exp \left\{ -\beta \hat{H}^{(0)} \right\} \right]$$

Grand-Canonical Free Energy

$$\mathcal{F}[j, j^*] = -\frac{1}{\beta} \ln \mathcal{Z}[j, j^*]$$

# Ginzburg-Landau Theory

Order parameter field in Matsubara transformation

$$\Psi_{i\alpha}(\omega_m) = \langle \hat{a}_{i\alpha}(\omega_m) \rangle = \beta \frac{\delta \mathcal{F}}{\delta j_{i\alpha}^*(\omega_m)}, \quad \omega_m = \frac{2\pi m}{\beta}$$

Effective action

$$\Gamma[\Psi, \Psi^*] = \mathcal{F}[j, j^*] - \frac{1}{\beta} \sum_{i, \omega_m, \alpha} [\Psi_{i\alpha}(\omega_m) j_{i\alpha}^*(\omega_m) + \Psi_{i\alpha}^*(\omega_m) j_{i\alpha}(\omega_m)]$$

Equations of motion for vanishing currents

$$\frac{\delta \Gamma}{\delta \Psi_{i\alpha}^*(\omega_m)} \Big|_{\Psi=\Psi_{\text{eq}}} = 0, \quad \frac{\delta \Gamma}{\delta \Psi_{i\alpha}(\omega_m)} \Big|_{\Psi=\Psi_{\text{eq}}} = 0$$

# Landau Theory

- Static case:

$$\Psi_{i\alpha}(\omega_m) = \Psi_\alpha \sqrt{\beta} \delta_{m,0}$$

- On-site effective potential

$$\Gamma(\Psi, \Psi^*) = \mathcal{F}_0 + \sum_{\alpha} B_{\alpha} |\Psi_{\alpha}|^2 + \sum_{\alpha_1, \alpha_2, \alpha_3, \alpha_4} A_{\alpha_1 \alpha_2 \alpha_3 \alpha_4} \Psi_{\alpha_1}^* \Psi_{\alpha_2}^* \Psi_{\alpha_3} \Psi_{\alpha_4}$$

- Landau coefficients

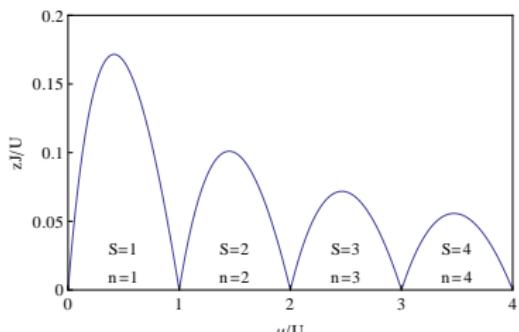
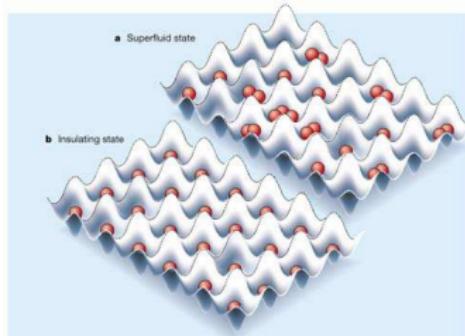
$$B_{\alpha} = \frac{1}{a_2^{(0)}(\alpha, 0)} - z J_{c,\alpha}$$

$$A_{\alpha_1 \alpha_2 \alpha_3 \alpha_4} = - \frac{a_4^{(0)}(\alpha_1, 0; \alpha_2, 0 | \alpha_3, 0; \alpha_4, 0)}{4 a_2^{(0)}(\alpha_1, 0) a_2^{(0)}(\alpha_2, 0) a_2^{(0)}(\alpha_3, 0) a_2^{(0)}(\alpha_4, 0)}$$

# Quantum Phase Boundary Without Zeeman effect

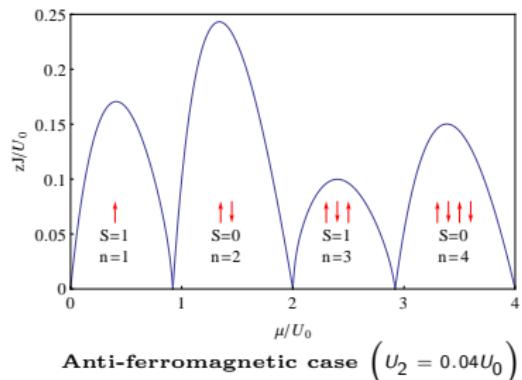
Location of phase transition:

$$J_c = \min_{\alpha} J_{c,\alpha}$$



Ferromagnetic case  $(U = U_0 + U_2)$

M. P. A. Fisher et al., Phys. Rev. B **40**, 546 (1989)

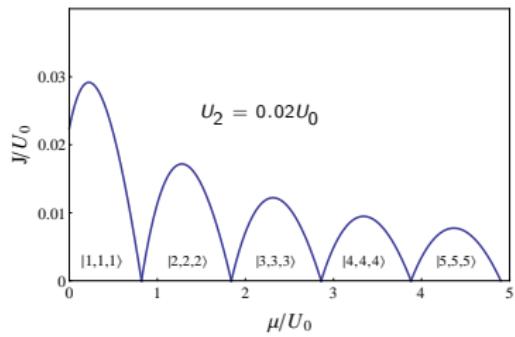


Anti-ferromagnetic case  $(U_2 = 0.04U_0)$

S. Tsuchiya et al., Phys. Rev. A **70**, 043628 (2004)

E. Demler et al., Phys. Rev. Lett. **88**, 163001 (2002)

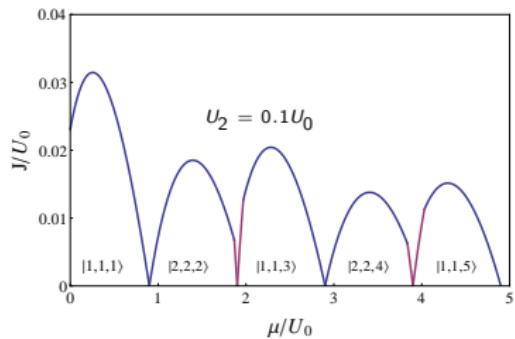
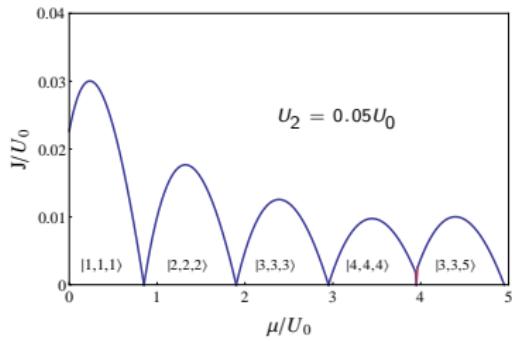
# Quantum Phase Boundary With Zeeman Effect ( $\eta = 0.2 U_0$ )



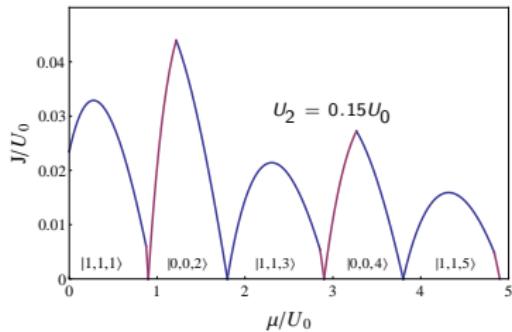
$$U_{2,\text{crit}}^{5 \rightarrow 3} = 0.044 U_0$$

$$U_{2,\text{crit}}^{3 \rightarrow 1} = 0.08 U_0$$

$$U_{2,\text{crit}}^{4 \rightarrow 2} = 0.057 U_0$$



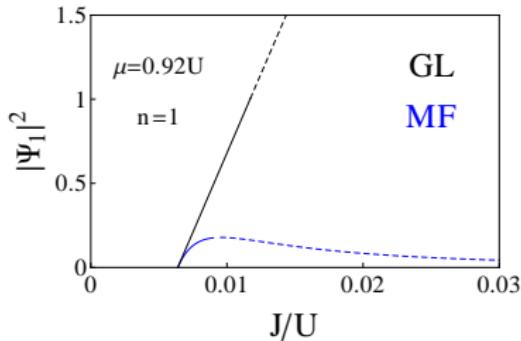
$$U_{2,\text{crit}}^{2 \rightarrow 0} = 0.133 U_0$$



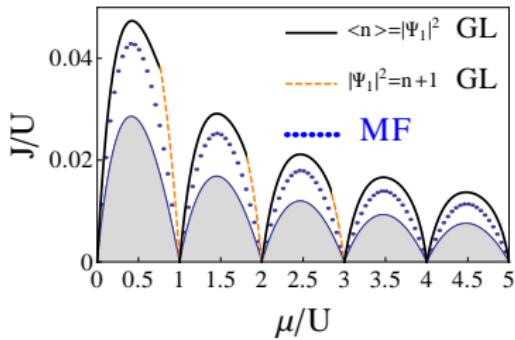
Blue (red) line corresponds to emerging non-vanishing spin-1 (spin-(-1)) component

# Validity Range

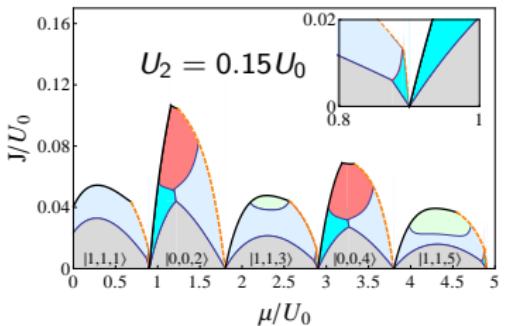
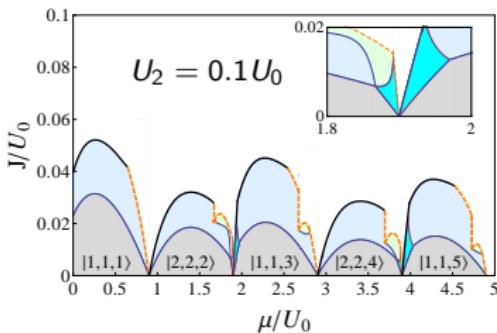
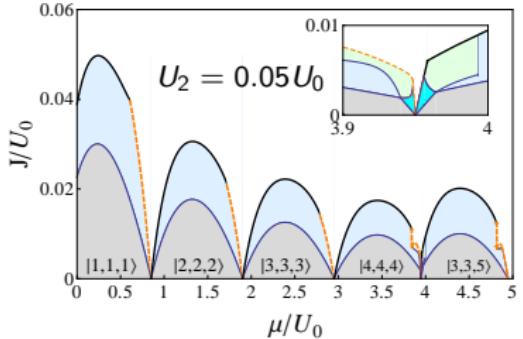
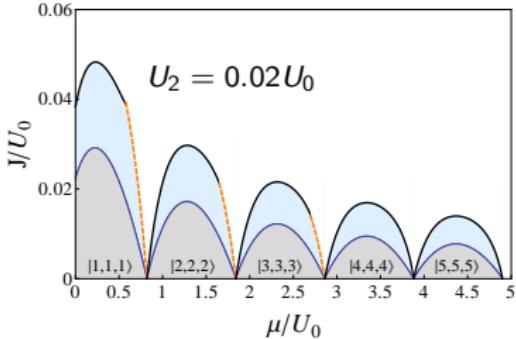
- Condensate density, e.g. ferromagnetic case ( $U = U_0 + U_2$ )



- Validity range



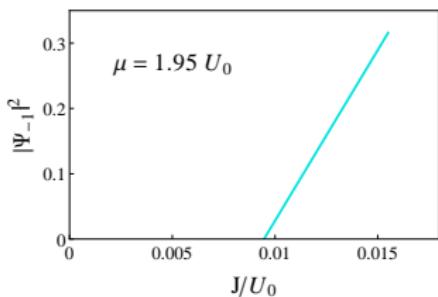
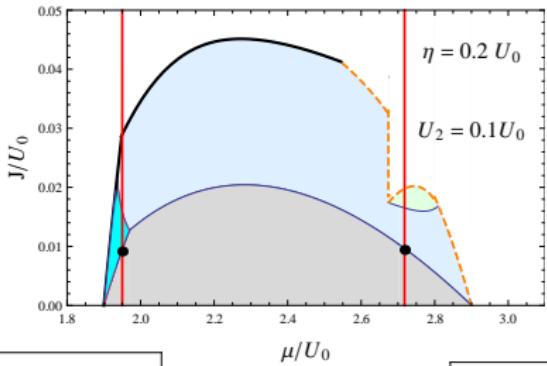
# Magnetic Superfluid Phases ( $\eta = 0.2 U_0$ )



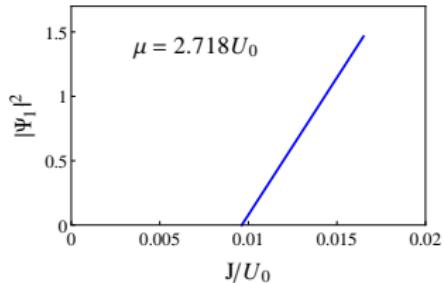
$$\Psi_1 \neq 0, \Psi_0 = \Psi_{-1} = 0 ; \Psi_{-1} \neq 0, \Psi_0 = \Psi_1 = 0 ; \Psi_1 \neq 0, \Psi_{-1} \neq 0, \Psi_0 = 0 ; \text{ and } \Psi_1 \neq 0, \Psi_{-1} \neq 0, \Psi_0 \neq 0$$

# Order of Mott Insulator - Superfluid Phase transitions

Second-order quantum phase transition

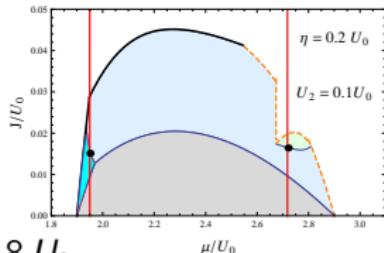


$$\Psi_{-1} \neq 0, \Psi_0 = \Psi_1 = 0$$

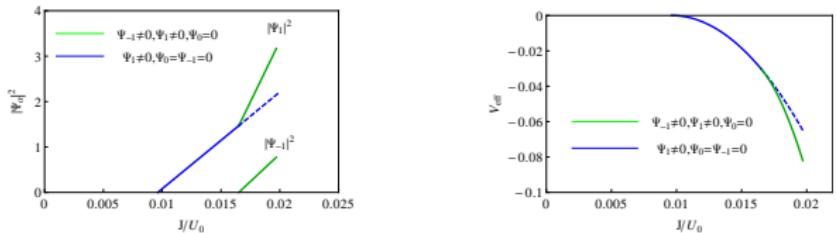


$$\Psi_1 \neq 0, \Psi_0 = \Psi_{-1} = 0$$

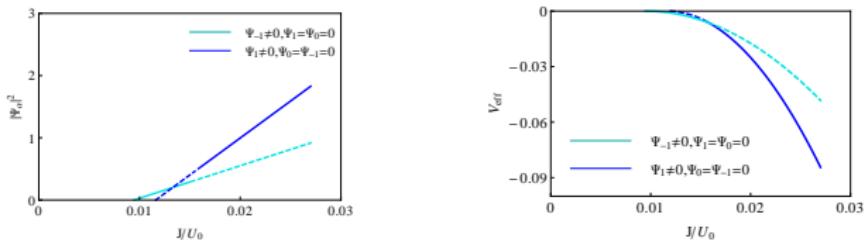
# Order of Transitions between Superfluid Phases



- Second-order:  $\mu = 2.718 U_0$



- First-order:  $\mu = 1.95 U_0$



Solid (dashed) lines correspond to solutions of minimal (not minimal) energy

# Summary and Outlook

- Phase diagram of different Mott phases with Zeeman effect at atomic limit
- New superfluid phases for anti-ferromagnetic interaction in presence of external magnetic field within range of validity of Ginzburg-Landau theory at zero temperature
- Second-order quantum phase transition from the Mott insulator to the superfluid phase under the effect of the magnetic field
- Both a first- and second-order phase transition can occur above the same Mott lobe in the superfluid phase
- **Outlook**
  - ① Different superfluid phases at non-zero temperature
  - ② Frustration in triangular optical lattice
  - ③ Superfluid phases of spin-1 bosons in superlattice

Thank you for your attention