Bose gases in confinements of finite size

Author: Möhnen, Til

AG Quantum Optics
FB Physik
TU Kaiserslautern

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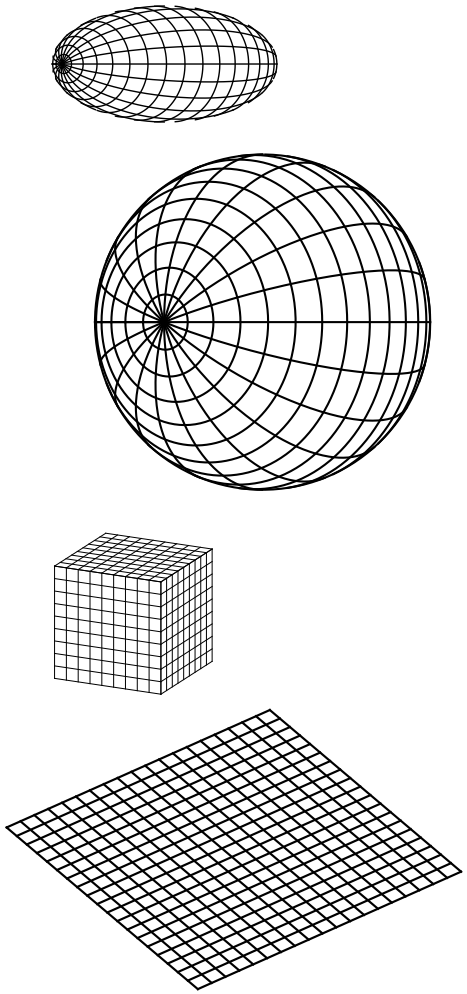
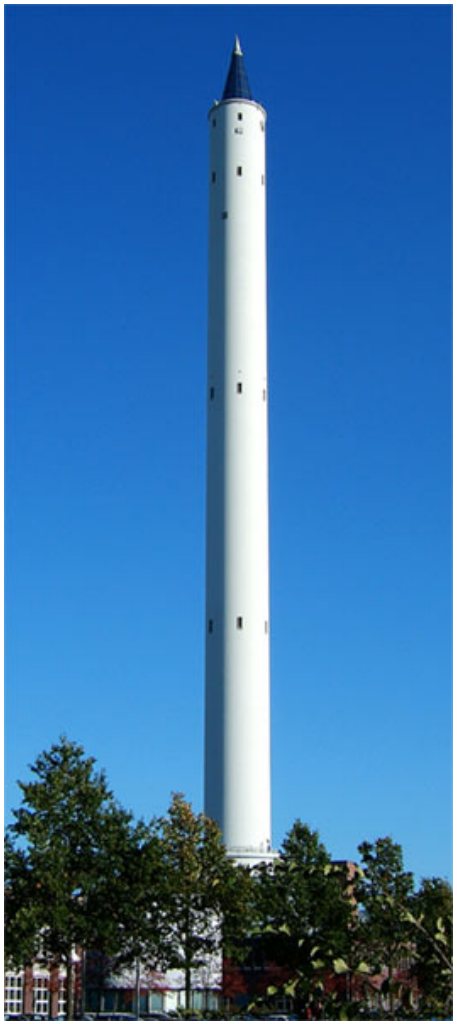
Overview

- **Experimental motivation**
  for investigating bubble-shaped gases
- **Thermodynamics of ideal Bose gas**
  on sphere
- **Including interaction**
  for arbitrary geometry
- **Experimental motivation**
  for investigating box-shaped gases
- **Cancelling UV divergencies**
  for 3D box
- **Outlook**
  for master's Thesis



Experimental motivation for investigating bubble-shaped gases

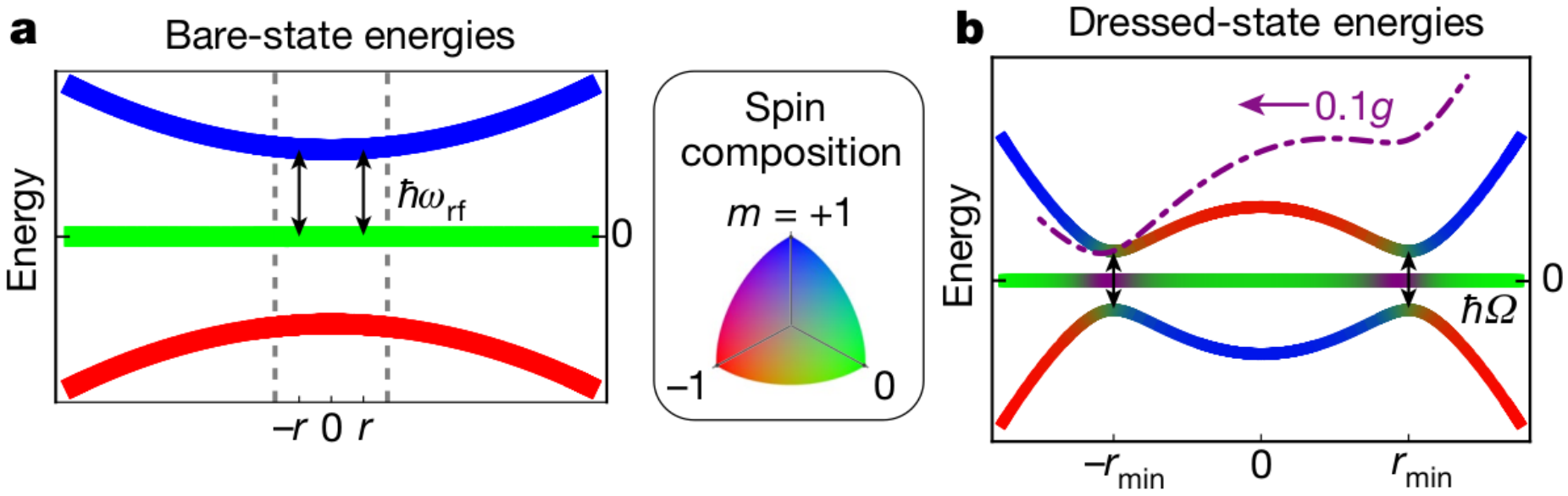


ZARM's Droptower in Bremen

NASA's Cold Atom Lab (CAL) at ISS

Generating potentials for neutral atoms

Zobay, O. & Garraway, B. M. *Two-dimensional atom trapping in field-induced adiabatic potentials*. Phys. Rev. Lett. 86 (2001)



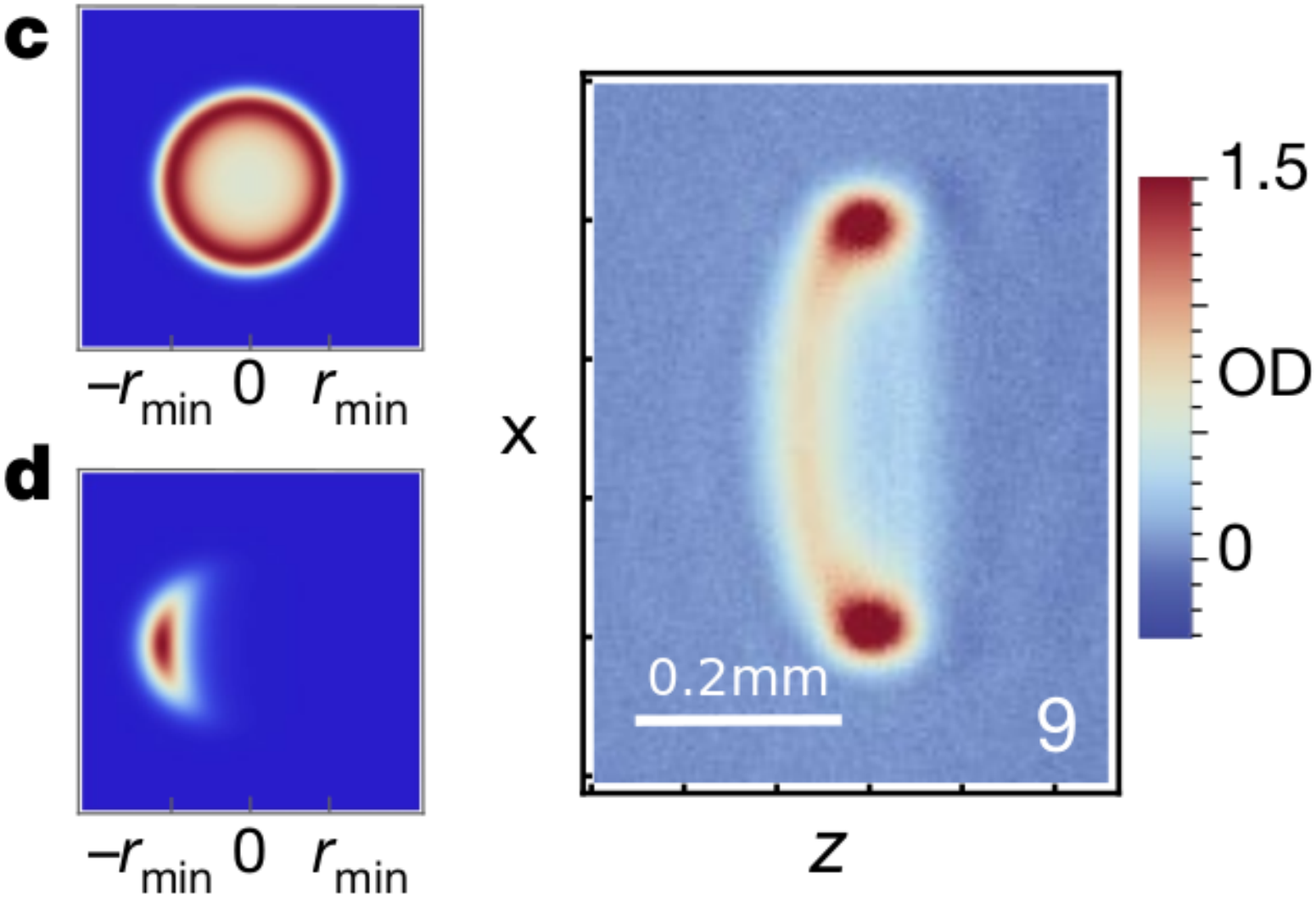
Lundblad N. et al. *Observation of ultracold atomic bubbles in orbital microgravity* Nature Vol.606 (2022)

$$U_m(r) = m g_F \mu_B \|B(r)\|$$

$$U_m(r) - U_m(0) \propto \|r\|^2$$

$$U_1^*(r) = \sqrt{(U_1(r) - \hbar\omega_{rf})^2 + (\hbar\Omega)^2}$$

Experimental results of CAL



Lundblad N. et al. *Observation of ultracold atomic bubbles in orbital microgravity* Nature Vol.606 (2022)

Ideal Bose gas on sphere

Single particle states

$$\frac{\hat{L}^2}{2mR^2} Y_{lm} = e_l Y_{lm}$$

$$e_l = \frac{\hbar^2 l(l+1)}{2mR^2}$$

Grand canonical ensemble

$$\Omega = (e_0 - \mu) N_0 + \frac{1}{\beta} \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \ln\left(1 - e^{-\beta(e_l - \mu)}\right)$$

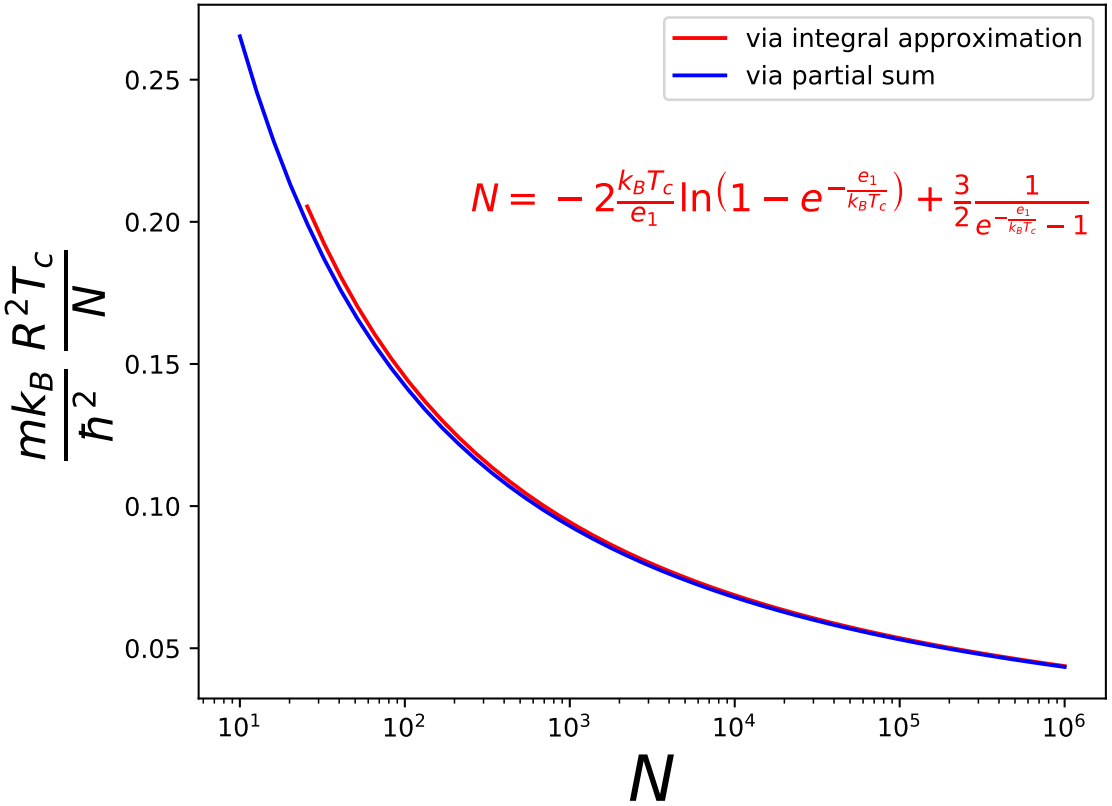
$$N \equiv -\left(\frac{\partial \Omega}{\partial \mu}\right)_{T,V} = N_0 + \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \frac{1}{e^{\beta(e_l - \mu)} - 1}$$

$$N_0 = \psi_0 \psi_0^* \ \wedge \ \frac{\partial \Omega}{\partial \psi_0} = 0 \ \Rightarrow \ N_0 = 0 \ \vee \ \mu = 0$$

evaluating series

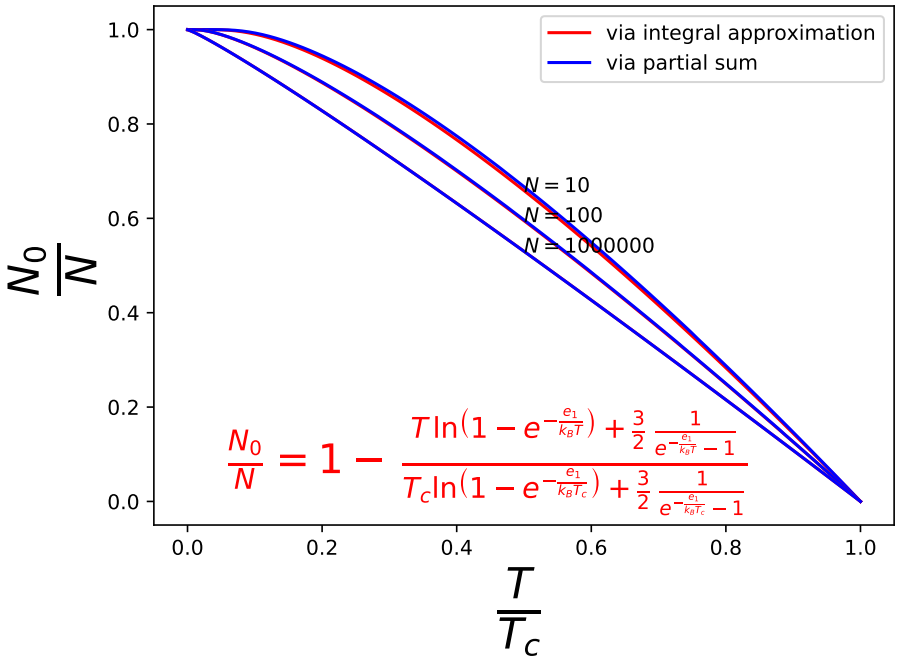
$$\sum_{l=1}^{\infty} \cdots \approx \begin{cases} \sum_{l=1}^{L} \cdots & \text{partial sum} \\ \int_{l}^{\infty} \cdots dl + \frac{1}{2}\left(\cdots|_{l=1} + \cdots|_{l=\infty}\right) & \text{integral approximation} \end{cases}$$

Phase transition



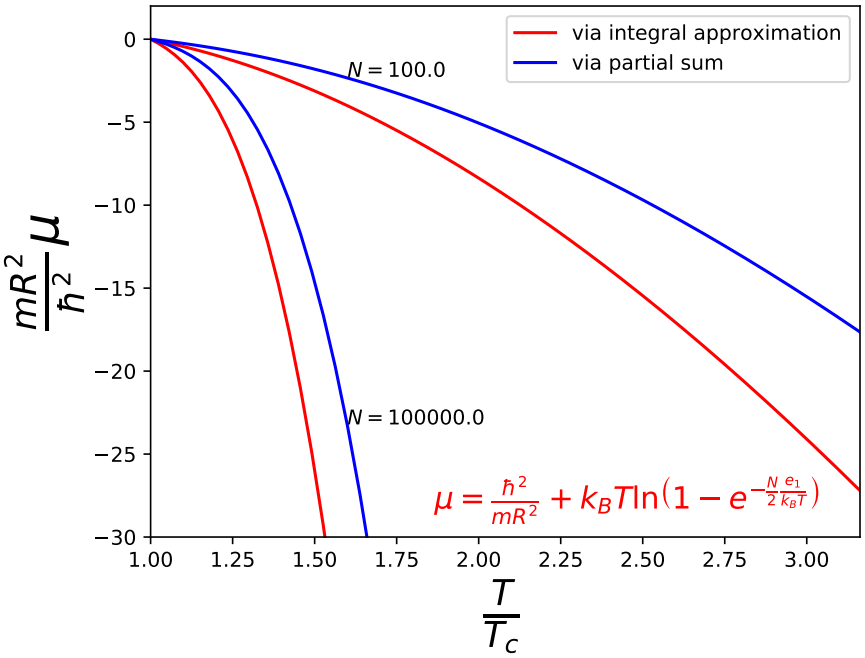
Ground state occupation

$T \leq T_c \Rightarrow \mu = 0$

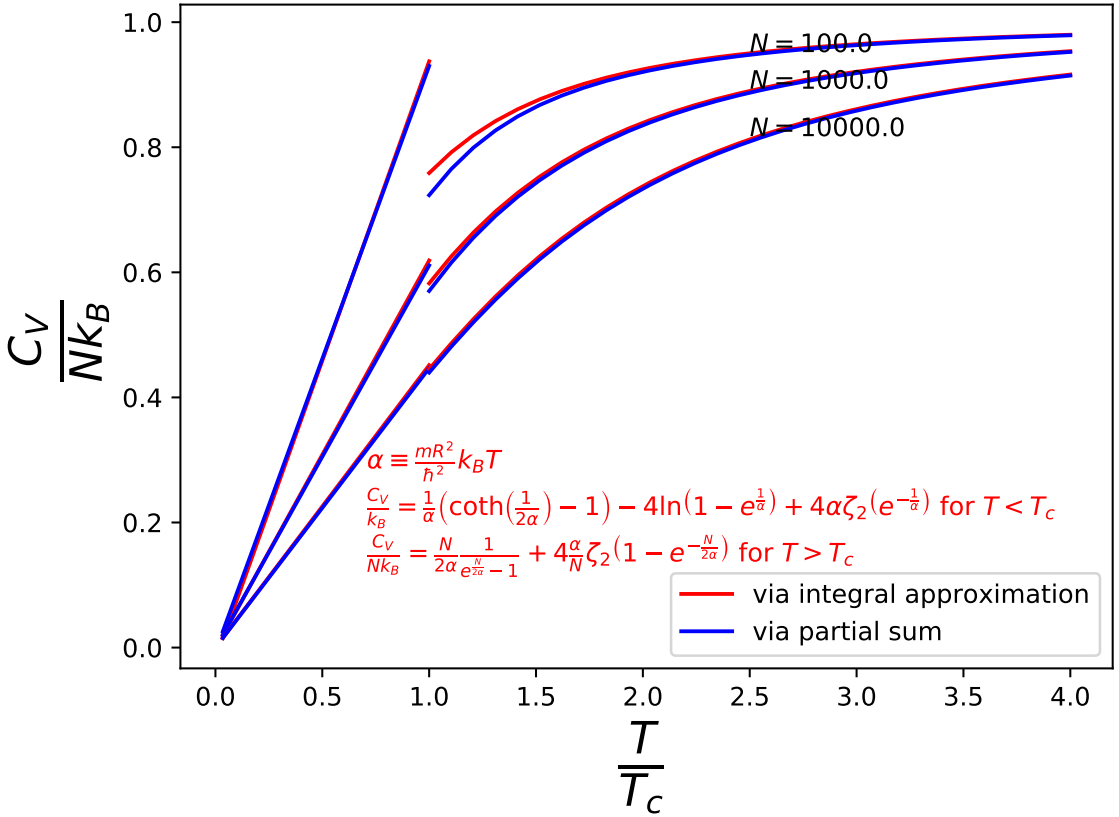


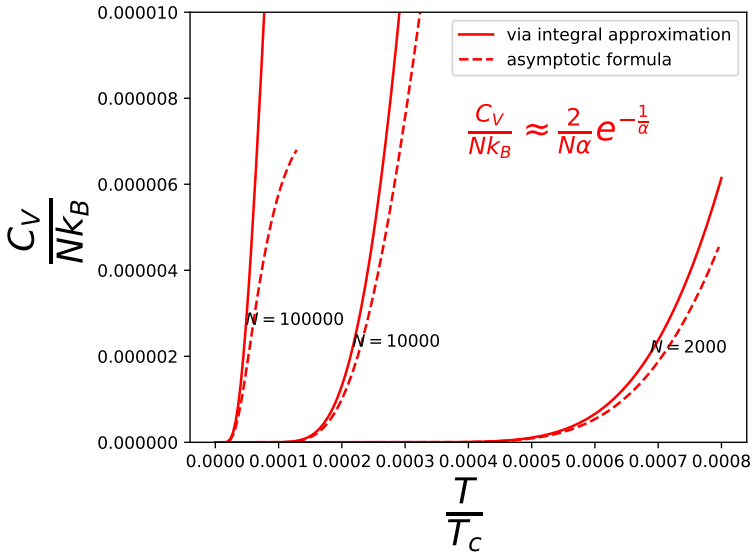
Chemical potential

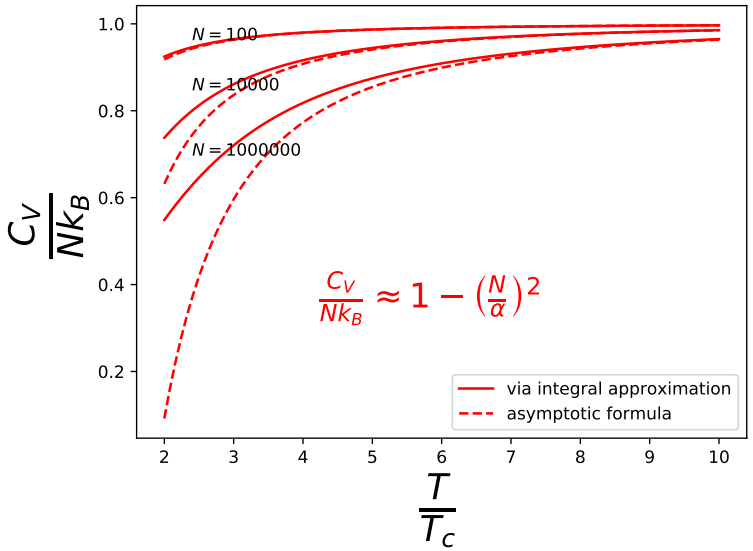
$T \geq T_c \Rightarrow N_0 = 0$



Heat capacity







Including weak interaction

$$\Omega \equiv -k_B T \ln(Z) \qquad Z \equiv \mathrm{Tr}\left(e^{-\frac{\hat{H}-\mu\hat{N}}{k_B T}}\right)$$

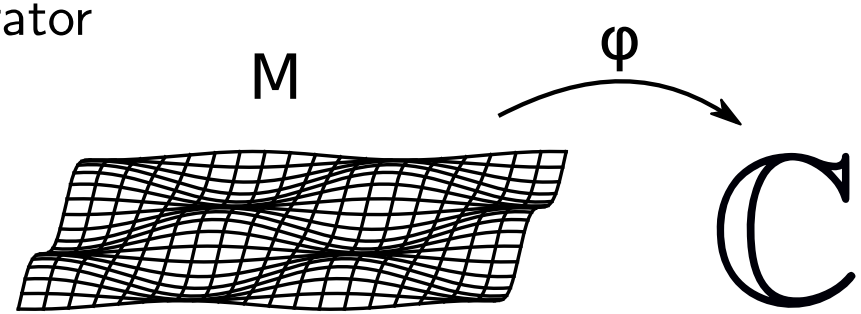
$$\hat{H} = \int_M \hat{a}^\dagger(r)\left(-\frac{\hbar^2}{2m}\Delta_{\mathrm{LB}}\right)\hat{a}(r)\ dr + \frac{1}{2}\int_M\int_M \hat{a}^\dagger(r)\hat{a}^\dagger(r')g\delta(r-r')\hat{a}(r)\hat{a}(r')\ dr\ dr'$$

$$\hat{N} = \int_M \hat{a}^\dagger(r)\hat{a}(r)\ dr$$

$$\left[\hat{a}(r), \hat{a}^\dagger(r')\right] = \delta(r-r') \quad \wedge$$

$$\left[\hat{a}(r), \hat{a}(r')\right] = 0 \quad \wedge \quad \left[\hat{a}^\dagger(r), \hat{a}^\dagger(r')\right] = 0$$

Laplace Beltrami operator



$$\Delta_{\mathrm{LB}}\varphi \equiv \frac{1}{\sqrt{|\det(g)|}}\nabla\cdot\left(\sqrt{|\det(g)|}g^{-1}\nabla\right)\varphi \qquad g\widehat{=}\text{ metric tensor}$$

Eigenvalue Problem $\Delta_{\mathrm{LB}}\varphi_\nu = -\frac{2m}{\hbar^2}E_\nu\varphi_\nu$

$$\varphi_0 = \text{const}$$
$$\Rightarrow \Delta_{\mathrm{LB}}\varphi_0 = 0\varphi_0$$

$$\Delta_{\mathrm{LB}}\varphi = \lambda\varphi$$
$$\Rightarrow \Delta_{\mathrm{LB}}\varphi^* = \lambda\varphi^*$$

$$\{\varphi_\nu\}\text{ complete and orthonormal} \xrightarrow{?} \varphi_{\nu_*} = \varphi_\nu^*$$

Plane: $\varphi_k(x) = \frac{1}{\sqrt{V}}e^{ik\cdot x} \quad \Rightarrow \quad k_* = -k$

Sphere: $\varphi_{l,m} = i^m Y_{lm} \quad \Rightarrow \quad (l,m)_* = l,-m$

Generalized Boguliubov theory

$$\hat{H} - \mu\hat{N}$$

$$= \cdots \int_M \hat{a}^\dagger(r)^2 \hat{a}(r)^2 \, dr$$

$$= \cdots \sum_{\nu\nu'\mu\mu'} \hat{b}_\nu^\dagger \hat{b}_{\nu'}^\dagger \hat{b}_\mu \hat{b}_{\mu'} \int_M \varphi_\nu^*(r)\varphi_{\nu'}^*(r)\varphi_\mu(r)\varphi_{\mu'}(r) \, dr$$

$$\approx \underbrace{N_0^0 \cdots \quad + N_0^{\frac{1}{2}} \cdots}_{\approx 0} \quad + N_0^1 \cdots \quad + N_0^{\frac{3}{2}} \cdots \quad + N_0^2 \cdots$$

$$\approx N_0\left(\frac{gN_0}{2V} - \mu\right) + \frac{gN_0}{2V} \sum_{\nu\neq 0}\left(\left(\hat{b}_\nu^\dagger \hat{b}_{\nu_*}^\dagger + \hat{b}_\nu \hat{b}_{\nu_*}\right) + \left(E_\nu - \mu + \frac{2gN_0}{V}\right)\hat{b}_\nu^\dagger \hat{b}_\nu\right)$$

$$= \cdots \quad + \hat{B}_\nu^\dagger \hat{B}_\nu \cdots \quad + \left(\hat{B}_\nu^\dagger \hat{B}_{\nu_*}^\dagger + \hat{B}_\nu \hat{B}_{\nu_*}\right) \underbrace{\cdots}_{=0}$$

$$= N_0\left(\frac{gN_0}{2V} - \mu\right) + \sum_{\nu\neq 0} \frac{\varepsilon_\nu - \sqrt{\varepsilon_\nu^2 + \left(\frac{gN_0}{V}\right)^2}}{2} + \varepsilon_\nu \hat{B}_\nu^\dagger \hat{B}_\nu$$

$$\hat{a}(r) = \sum_\nu \hat{b}_\nu \varphi_\nu(r)$$

$$\hat{b}_0 |N_0 \cdots\rangle = \sqrt{N_0}|N_0 - 1 \cdots\rangle \approx \sqrt{N_0}|N_0 \cdots\rangle$$

$$\int_M \varphi_0 \cdots dr = \varphi_0 \int_M \cdots dr$$

$$\int_M \varphi_\nu \varphi_{\nu'} \, dr = \int_M \varphi_{\nu_*}^* \varphi_{\nu'} \, dr = \delta_{\nu'\nu_*}$$

$$\hat{b}_\nu = u_\nu \hat{B}_\nu + v_\nu \hat{B}_{\nu_*}^\dagger$$

$$\left[\hat{B}_\nu, \hat{B}_\mu^\dagger\right] = \delta_{\mu\nu} \qquad \left[\hat{B}_\nu, \hat{B}_\mu\right] = 0$$

$$\left.\begin{matrix} u_\nu \\ v_\nu \end{matrix}\right\} = \pm\sqrt{\pm\frac{1}{2} + \frac{1}{2}\sqrt{1 + \left(\frac{gN_0}{V\varepsilon_\nu}\right)^2}}$$

$$\varepsilon_\nu = \sqrt{\left(E_\nu - \mu + 2\frac{gN_0}{V}\right)^2 - \left(\frac{gN_0}{V}\right)^2}$$

Effective grand-canonical potential

$$\Omega_{\text{eff}} = -k_B T \ln(Z) \approx N_0 \left(\frac{gN_0}{2V} - \mu \right) + \sum_{\nu \neq 0} \frac{\varepsilon_\nu - \sqrt{\varepsilon_\nu^2 + \left(\frac{gN_0}{V}\right)^2}}{2} + k_B T \sum_{\nu \neq 0} \ln\left(1 - e^{-\frac{\varepsilon_\nu}{k_B T}}\right)$$

$$\varepsilon_\nu = \sqrt{\left(E_\nu - \mu + 2\frac{gN_0}{V}\right)^2 - \left(\frac{gN_0}{V}\right)^2} \mathrel{\widehat{=}} \text{Bogoliubov dispersion}$$

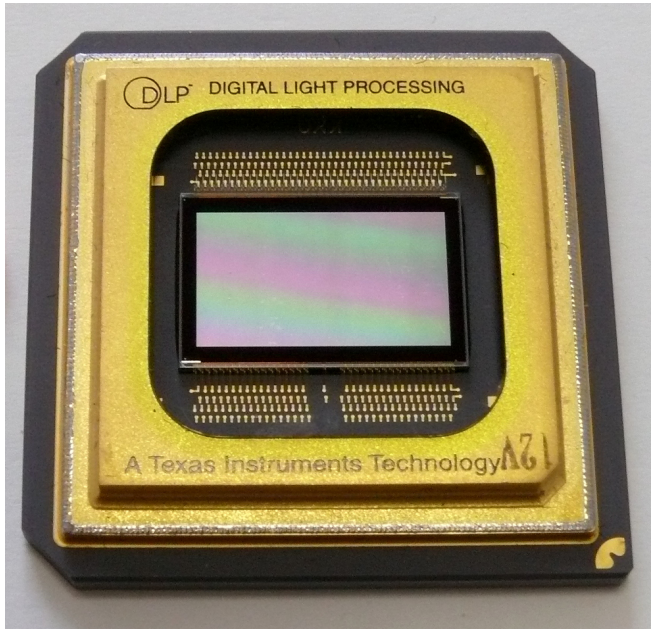
Elimination of ground-state occupation

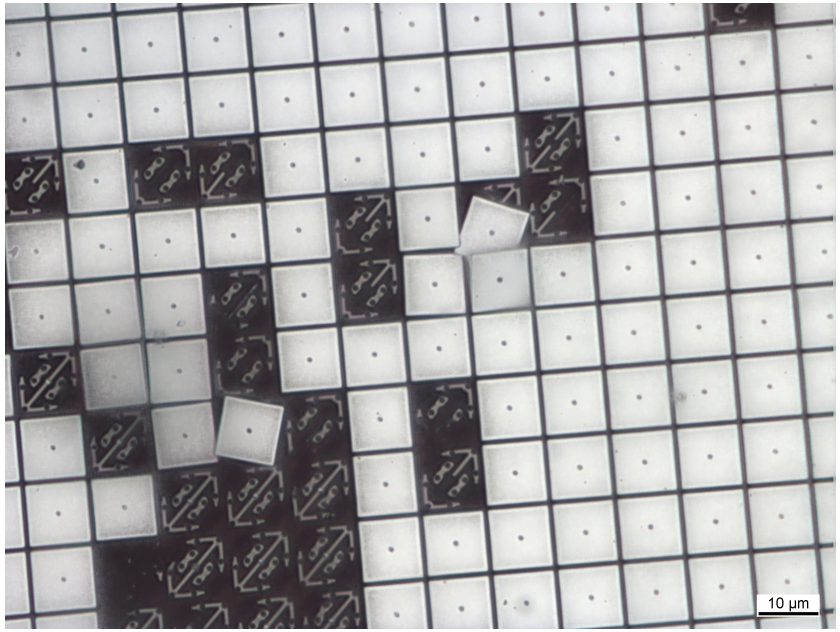
$$\frac{\partial \Omega_{\text{eff}}}{\partial N_0} \stackrel{!}{=} 0 \quad \Rightarrow \quad N_0 = \frac{V}{g}\mu + \sum_{\nu \neq 0} \left(1 - \frac{2E_\nu + \mu}{\sqrt{E_\nu (E_\nu + 2\mu)}} \left(\frac{1}{2} + \frac{1}{e^{\frac{\sqrt{E_\nu (E_\nu + 2\mu)}}{k_B T}} - 1} \right) \right) + \cdots$$

$$\Omega \approx -\frac{V}{2g}\mu^2 + \sum_{\nu \neq 0} \frac{\sqrt{E_\nu (E_\nu + 2\mu)} - E_\nu - \mu}{2} + k_B T \sum_{\nu \neq 0} \ln\left(1 - e^{-\frac{\sqrt{E_\nu (E_\nu + 2\mu)}}{k_B T}}\right)$$

Experimental motivation for investigating box-shaped gases

Digital micro-mirror device (DMD)





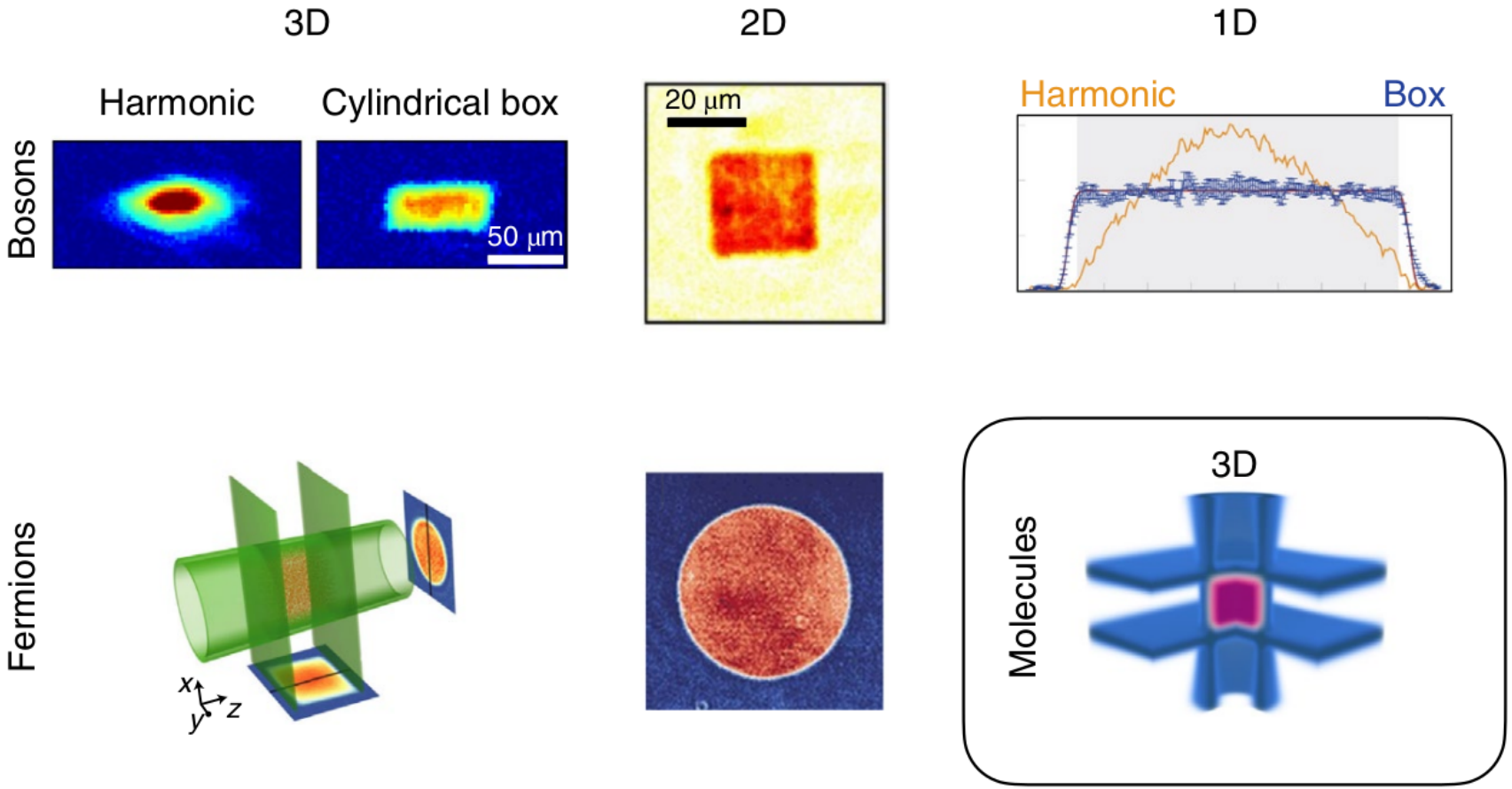
- number of pixels: $\sim 10^6$
- pixel size: $\sim 10\mu m$
- refreshrate: $\sim$ 10kHz

first publication using this device:

Gaunt, A. L., Schmidutz, T. F., Gotlibovych, I., Smith, R. P. & Hadzibabic, Z. *Bose–Einstein condensation of atoms in a uniform potential*. Phys. Rev. Lett. 110, 200406 (2013)

Experimental realization so far

Varying trap dimensionality and particle type

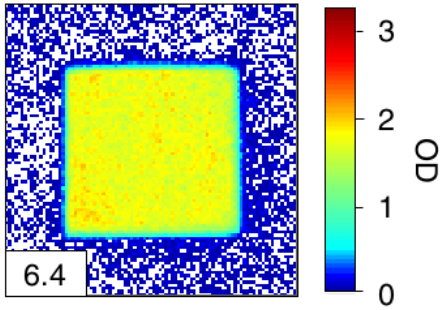


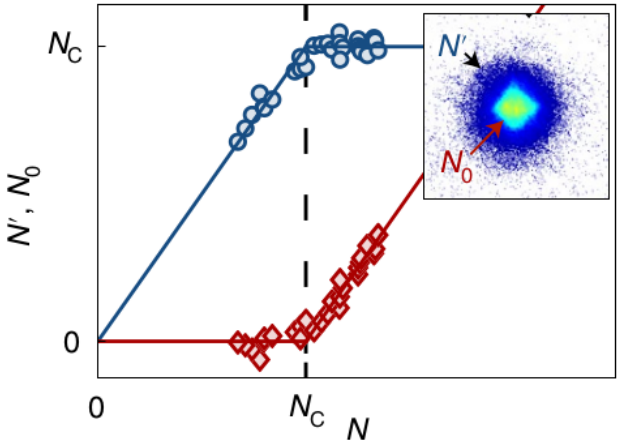
Navon, N., Smith, R.P. & Hadzibabic, Z. *Quantum gases in optical boxes*. Nat. Phys. 17 (2021)

Experimental success so far

Varying degree of perturbation

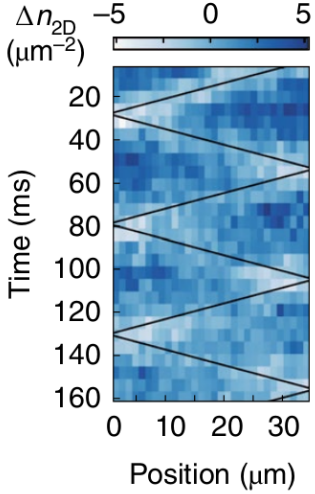
Bose-Einstein condensation

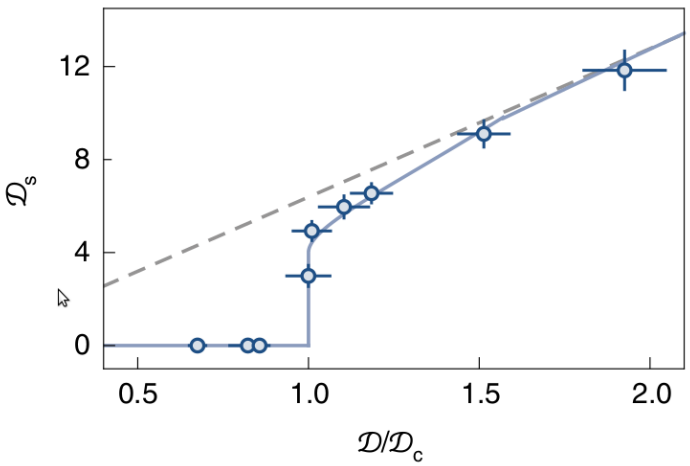




Navon, N., Smith, R.P. & Hadzibabic, Z. *Quantum gases in optical boxes*. Nat. Phys. 17 (2021)

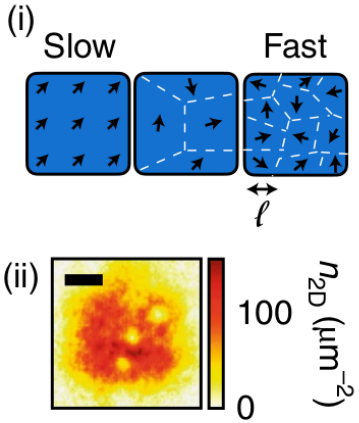
Sound in BKT superfluid

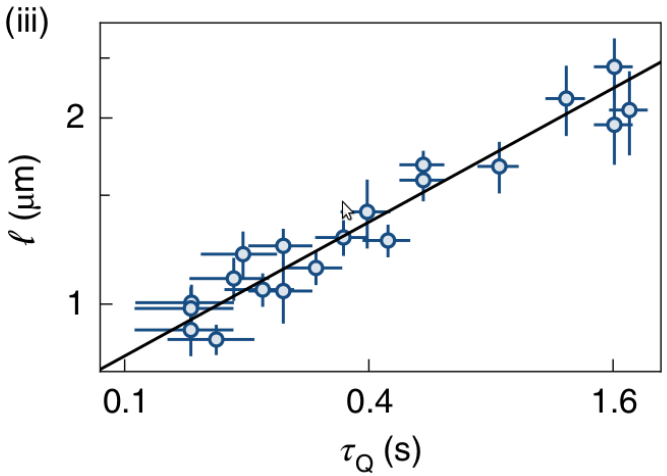




Ville, J. L. et al. *Sound propagation in a uniform superfluid two-dimensional Bose gas*. Phys. Rev. Lett. 121, 145301 (2018)

Kibble-Zurek phenomena





Navon, N., Gaunt, A. L., Smith, R. P. & Hadzibabic, Z. *Critical dynamics of spontaneous symmetry breaking in a homogeneous Bose gas*. Science 347 (2015)

Cancelling ultraviolet divergencies for 3D box

$$E_k = \frac{\hbar^2 \|k\|^2}{2m}$$

Regularization of quantum fluctuations

$$\frac{1}{2} \underbrace{\sum_{k \in \frac{2\pi}{L}\mathbb{Z}^3 \setminus 0}}_{\approx \left(\frac{L}{2\pi}\right)^3 \int_{\mathbb{R}^3} dk = \left(\frac{L}{2\pi}\right)^3 4\pi \lim_{\Lambda\to\infty} \int_0^\Lambda dk\ k^2} \left(\frac{\hbar\|k\|}{\sqrt{2m}} \sqrt{\frac{\hbar^2\|k\|^2}{2m} + 2\mu} - \frac{\hbar^2\|k\|^2}{2m} - \mu \right)$$

$$\approx \frac{L^3}{(2\pi)^2} \lim_{\Lambda\to\infty} \left(\frac{1}{5}\frac{\hbar^2}{2m}\Lambda^5 + \frac{1}{3}\mu\Lambda^3 - \frac{1}{2}\mu^2 \frac{2m}{\hbar^2}\Lambda + \frac{4}{15}\mu \left(\frac{\hbar}{\sqrt{4m\mu}}\right)^{-3} - \left[\frac{\hbar^2}{2m}\frac{1}{5}k^5 + \mu\frac{1}{3}k^3\right]_0^\Lambda \right)$$

$$= \frac{8}{15\pi^2}\frac{m^{\frac{3}{2}}}{\hbar^3} L^3 \mu^{\frac{5}{2}} - \frac{1}{4\pi^2}\frac{m}{\hbar^2} L^3 \mu^2 \lim_{\Lambda\to\infty} \Lambda$$

Renormalization of interaction strength

$$\sigma = \frac{1}{4\pi\left(\frac{\hbar^2}{mg} + (2\pi)^{-3}\int\limits_{\mathbb{R}^3}\frac{1}{\|k\|^2}\,dk\right)^2} \overset{!}{=} \frac{1}{4\pi\left(\frac{\hbar^2}{mg_R}\right)^2}$$

$$\frac{1}{g} = \frac{1}{g_R} - \frac{m}{\hbar^2}(2\pi)^{-3}\underbrace{\int\limits_{\mathbb{R}^3}\frac{1}{\|k\|^2}\,dk}_{=\lim\limits_{\Lambda\to\infty}\int\limits_0^\Lambda\frac{k^2}{k^2}\,dk=4\pi\lim\limits_{\Lambda\to\infty}\Lambda}$$

Finite grand-canonical potential

$$\Omega \approx -\frac{L^3}{2}\mu^2\left(\frac{1}{g_R} - \frac{1}{2\pi^2}\frac{m}{\hbar^2}\lim_{\Lambda\to\infty}\Lambda\right) + \frac{8}{15\pi^2}\frac{m^{\frac{3}{2}}}{\hbar^3}L^3\mu^{\frac{5}{2}} - \frac{1}{4\pi^2}\frac{m}{\hbar^2}L^3\mu^2\lim_{\Lambda\to\infty}\Lambda + \cdots$$

$$= -\frac{1}{2g_R}L^3\mu^2 + \frac{8}{15\pi^2}\frac{m^{\frac{3}{2}}}{\hbar^3}L^3\mu^{\frac{5}{2}} + k_BT\sum_{k\in\frac{2\pi}{L}\mathbb{Z}^3\setminus 0}\ln\left(1 - e^{-\frac{\frac{\hbar\|k\|}{\sqrt{2m}}\sqrt{\frac{\hbar^2\|k\|^2}{2m}+2\mu}}{k_BT}}\right)$$

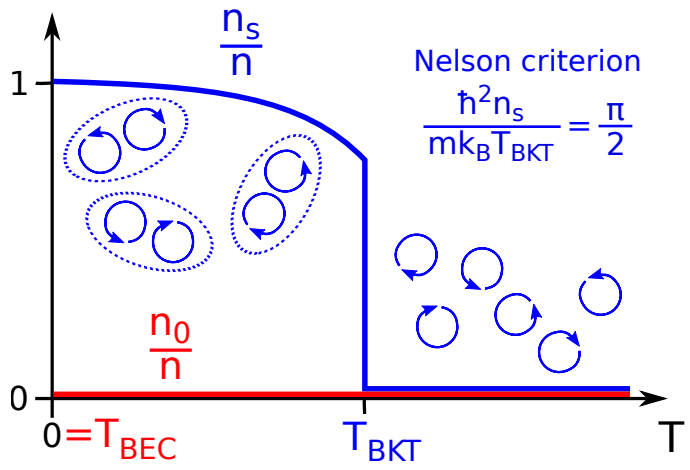
Outlook for master's thesis

Finite-size effects of 2D Box

Derivation of grand-canonical potential analogous to 3D box except...

- Infrared divergency in addition to ultraviolett divergency
- Scattering theory more complicated
- Include superfluidity by considering additional contribution in Hamiltonian

Thermodynamic limit



Finite system