Weakly Interacting Bose Gas on Curved Manifolds

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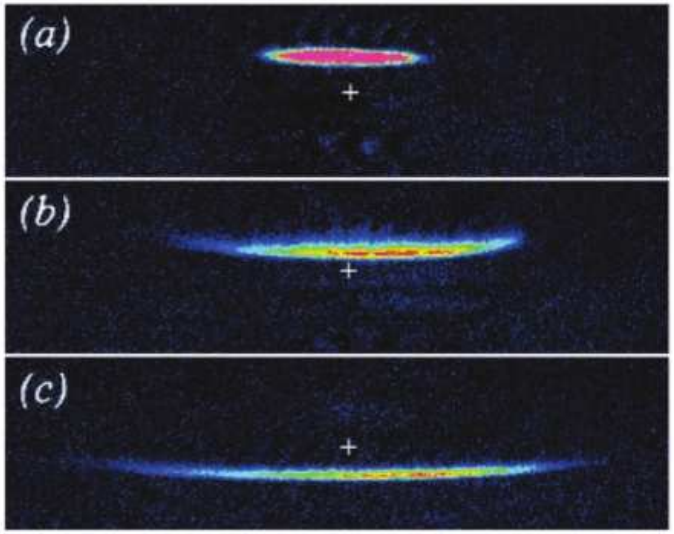
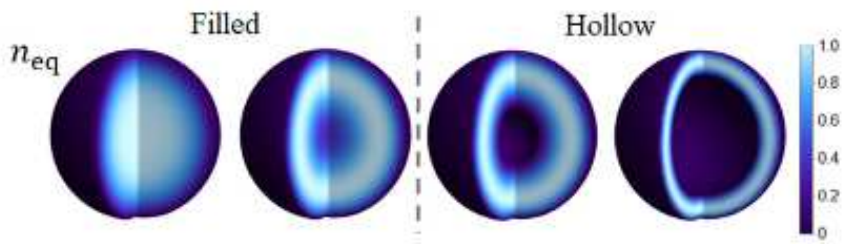
Experiment: bubble trap



Y. Colombe et al., EPL **67**, 593 (2004)

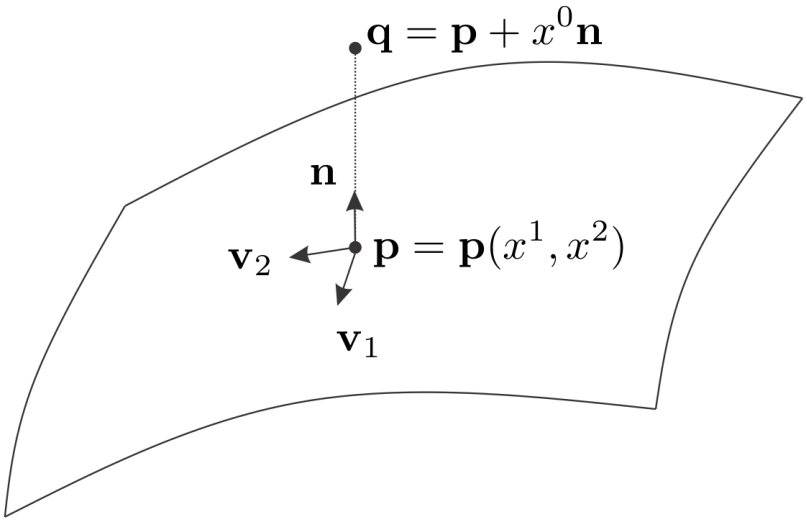
Theory: dimensional crossover



K. Sun et al., PRA **98**, 013609 (2018)

1 Manifold

- **Gaussian normal coordinate system:**



- **Metric:**

$$G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & & \\ 0 & & g \end{pmatrix}$$

- **Potential confinement:** $$V = \frac{M}{2}\,\omega^2\,(x^0)^2$$

2 Reducing Dimensionality

- **Particle number:** $N = \int dV \, |\Psi(x^0, x^1, x^2)|^2$

- **Energy:**

$$E = \int dV \, \Psi^* \left[-\frac{\hbar^2}{2M} \Delta + \frac{M}{2} \omega^2 (x^0)^2 + \frac{1}{2} g |\Psi|^2 - \mu \right] \Psi$$

- **Ansatz:**

$$\Psi(x^0, x^1, x^2) = \frac{e^{-(x^0)^2/2\sigma^2(x^1,x^2)}}{\sqrt[4]{\pi}\sqrt{\sigma(x^1, x^2)}} \psi(x^1, x^2)$$

- **Integrate variable x^0 perpendicular to manifold**

- **Extremize energy with respect to $\psi^*(x^1, x^2)$ and $\sigma(x^1, x^2)$**

Inspired by L. Salasnich et al., PRA **65**, 043614 (2002)

3 Quasi Two Dimensions

- **Particle number:** $$N = \int d^2x \, \sqrt{\det g} \, |\psi(x^1, x^2)|^2$$

- **Two-dimensional Gross-Pitaevskii equation:**

$$\left[-\frac{\hbar^2}{2M} \Delta_{\mathrm{LB}} + V_{\mathrm{eff}}(x^1, x^2) + \frac{\hbar^2}{4M\sigma^2} + \frac{M\omega^2\sigma^2}{4} + \frac{g|\psi|^2}{\sqrt{2\pi}\sigma} \right] \psi = \mu \, \psi$$

- **Laplace-Beltrami operator:**

$$\Delta_{\mathrm{LB}} = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i} \left(\sqrt{\det g} \, g^{ij} \frac{\partial}{\partial x^j} \right)$$

- **Width:**

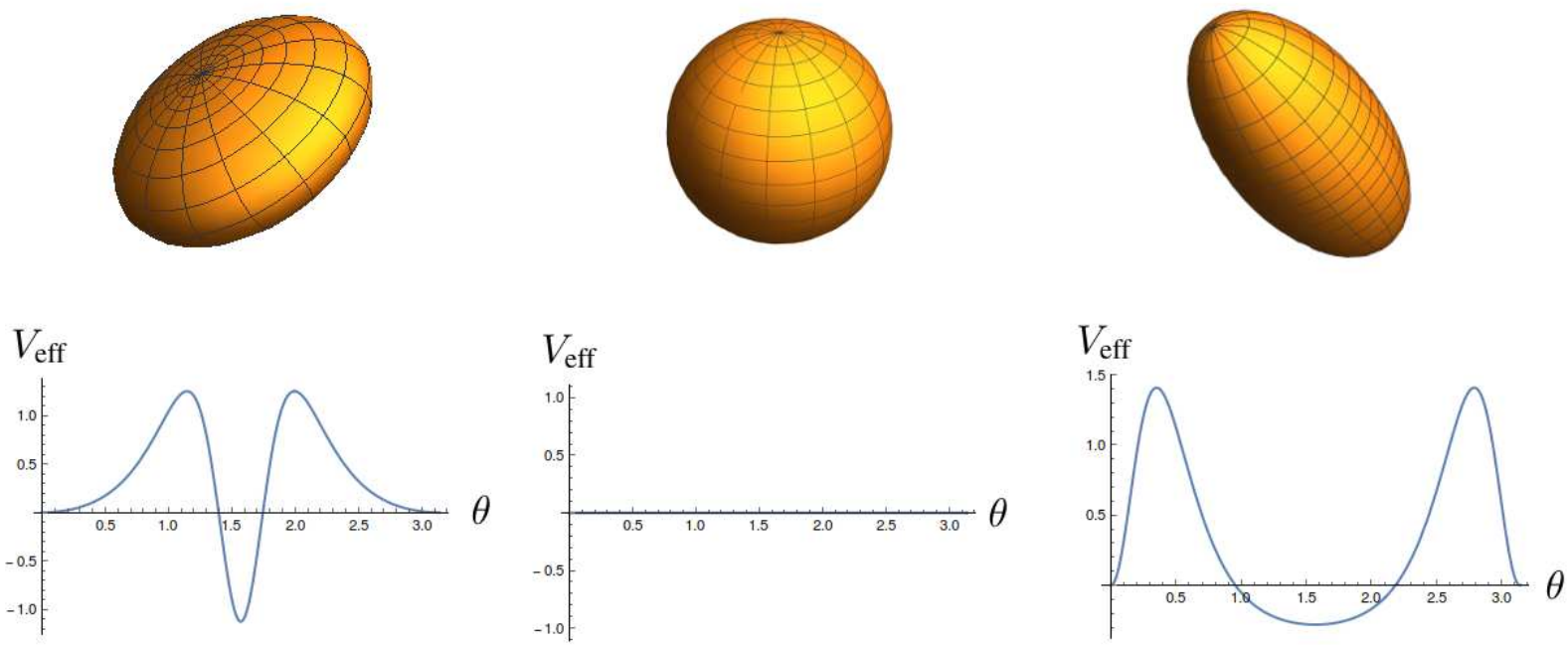
$$\frac{\sigma^4}{\sigma_{\mathrm{osc}}^4} = 1 + \frac{gM}{\sqrt{2\pi}\hbar^2} \sigma |\psi|^2 \,, \qquad \sigma_{\mathrm{osc}} = \sqrt{\frac{\hbar}{M\omega}}$$

4 Effective Potential

- **General expression:**

$$V_{\text{eff}}(x^1, x^2) = \frac{\hbar^2}{4M}\left[\frac{1}{2}\left(\frac{\partial \ln\sqrt{\det g}}{\partial x^0}\right)^2 + \frac{\partial^2 \ln\sqrt{\det g}}{\partial {x^0}^2}\right]\Bigg|_{x^0=0}$$

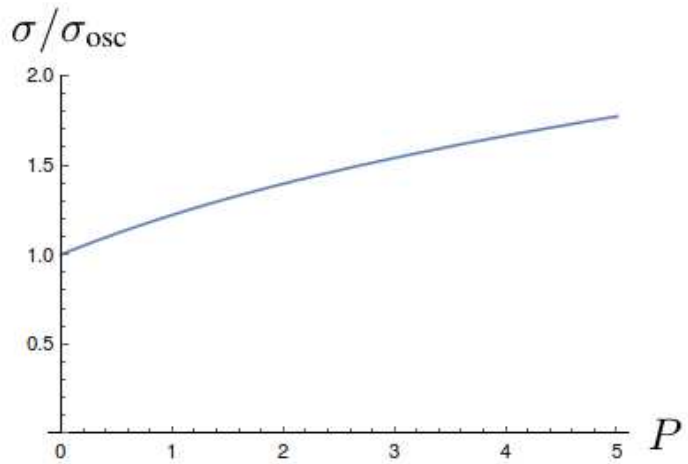
- **Examples:**

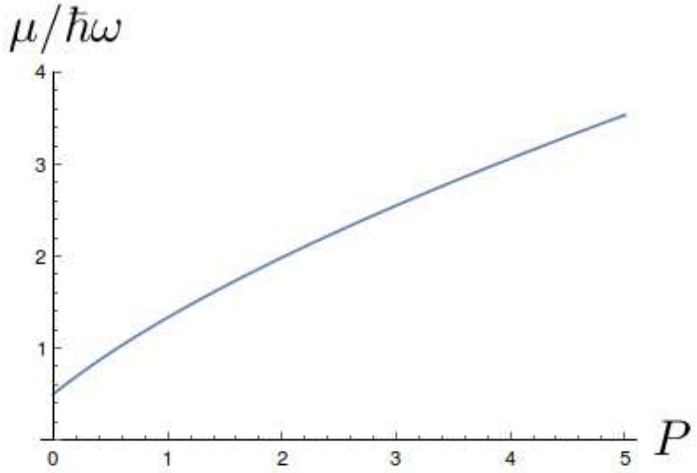


5 Equilibrium on Quasi Sphere

- **Wave function from normalization:** $\psi = \sqrt{\dfrac{N}{4\pi R^2}}$

- **Width:** **Equation of state:**





$$\left(\frac{\sigma}{\sigma_{\rm osc}}\right)^4 = 1 + P\,\frac{\sigma}{\sigma_{\rm osc}} \qquad \frac{\mu}{\hbar\omega} = \frac{1}{4}\left(\frac{1}{\sigma^2/\sigma_{\rm osc}^2} + \frac{\sigma^2}{\sigma_{\rm osc}^2}\right) + \frac{P}{\sigma/\sigma_{\rm osc}}$$

- **Dimensionless interaction strength:** $P = \dfrac{a_{\rm s}\sigma_{\rm osc}N}{\sqrt{2\pi}R^2}$

$$a_{\rm s} = 100\,a_{\rm B}, \sigma_{\rm osc} = 1\,\mu{\rm m}, R = 100\,\mu{\rm m}, N = 10^5 \quad \Longrightarrow \quad P = 2\times10^{-2}$$

6 Low-Lying Excitations on Quasi Sphere

- **Action:**

$$S = \int dt \int dV\, \Psi^* \left[i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2M} \Delta - \frac{M}{2} \omega^2 (x^0)^2 - \frac{1}{2} g |\Psi|^2 \right] \Psi$$

- **Ansatz:**

$$\Psi(x^0, x^1, x^2, t) = \frac{\exp\left\{ -\frac{1}{2} \left[\frac{1}{\sigma^2(x^1, x^2, t)} + iB(x^1, x^2, t) \right] (x^0)^2 \right\}}{\sqrt[4]{\pi} \sqrt{\sigma(x^1, x^2, t)}} \psi(x^1, x^2, t)$$

- **Linear stability analysis**
- **Lower frequencies:**

$$\Omega_l' = \frac{\hbar}{2MR^2} l(l+1) + \mathcal{O}(P)$$

Higher frequencies:

$$\Omega_l'' = 2\omega - \frac{\hbar}{2MR^2} l(l+1) + \mathcal{O}(P)$$

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Carlos Sa de Melo (Atlanta, USA): *Many-body Theory, BEC-BCS Crossover*

Jean Zinn-Justin (Paris, France): *Quantum Field Theory, Large-N Technique*

Victor Dotsenko (Paris, France): *Random Matrix Theory, Replica Trick*

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