Bose-Einstein Condensation on Curved Manifolds

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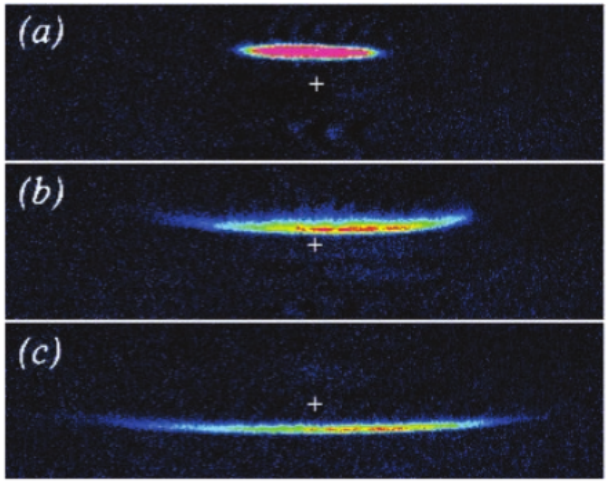
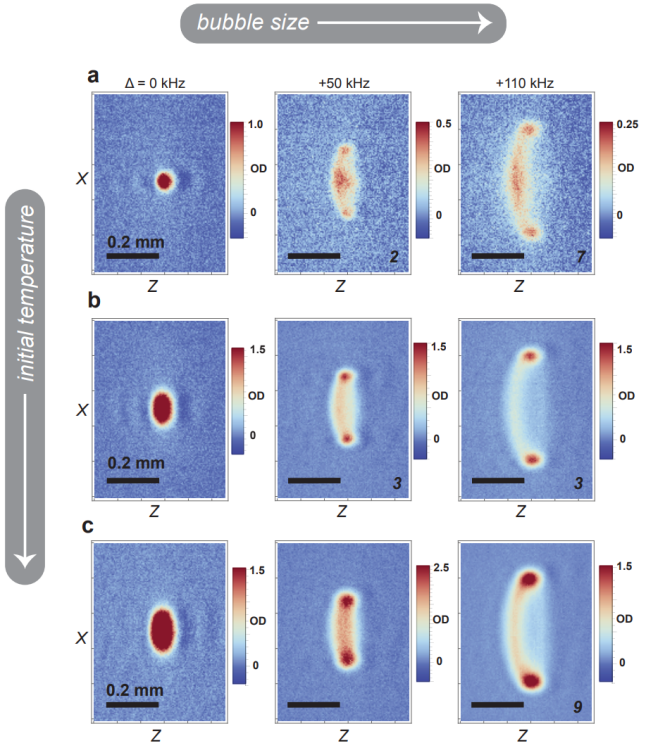
Bubble-trap





O. Zobay and B. M. Garraway, PRL **86** 1195 (2001); Y. Colombe, E. Knyazchyan, O. Morizot, B. Mercier, V. Lorentand, and H. Perrin, EPL **67** 593 (2004).

R. A. Carollo, D. C. Aveline, B. Rhyno, S. Vishveshwara, C. Lannert, J. D. Murphree, E. R. Elliott, J. R. Williams, R. J. Thompson, N. Lundblad, arXiv:2108.05880.

BEC on a Curved Manifold

Bose-Einstein Condensation on Curved Manifolds

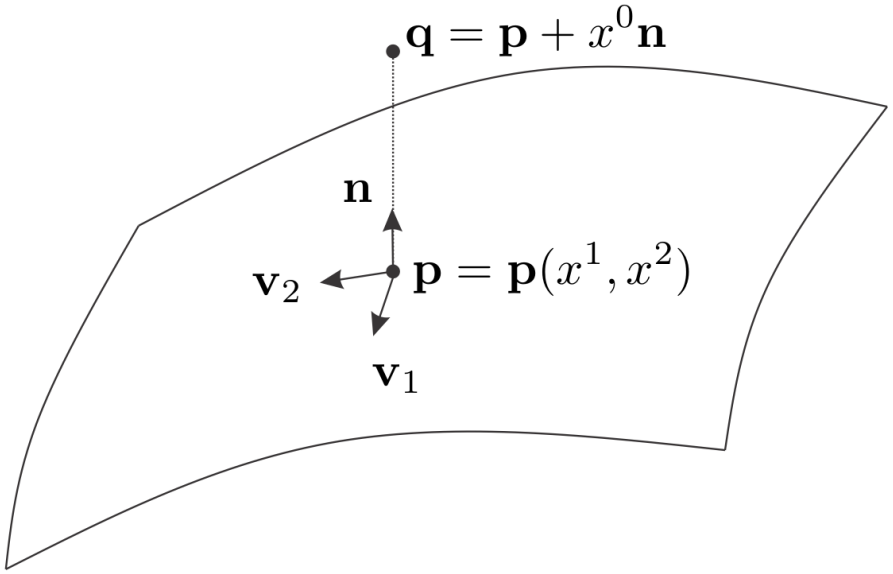
Natália S. Móller, F. Ednilson A. dos Santos, Vanderlei S. Bagnato, and Axel Pelster

Abstract. Here we describe a weakly interacting Bose gas on a curved smooth manifold, which is embedded in the three-dimensional Euclidean space. To this end we start by considering a harmonic trap in the normal direction of the manifold, which confines the three-dimensional Bose gas in the vicinity of its surface. Following the notion of dimensional reduction as outlined in [L. Salasnich et al., Phys. Rev. A **65**, 043614 (2002)], we assume a large enough trap frequency so that the normal degree of freedom of the condensate wave function can be approximately integrated out. In this way we obtain an effective condensate wave function on the quasi-two-dimensional surface of the curved manifold, where the thickness of the cloud is determined self-consistently. For the particular case when the manifold is a sphere, our equilibrium results show how the chemical potential and the thickness of the cloud increase with the interaction strength. Furthermore, we determine within a linear stability analysis the low-lying collective excitations together with their eigenfrequencies, which turn out to reveal an instability for attractive interactions.

N. Móller, F. E. A. dos Santos, V. S. Bagnato, A. Pelster, NJP (2020).

Manifold

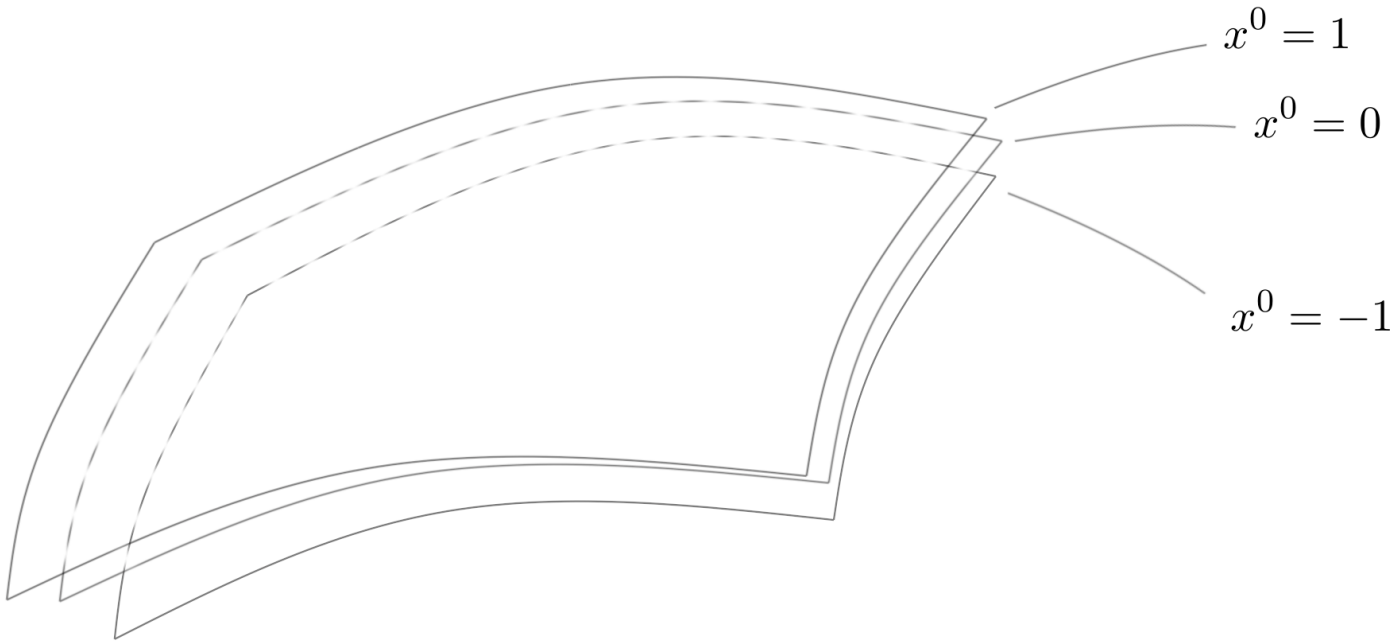
Gaussian normal coordinate system:



Metric: $g(x^1,x^2) = g(x^0 = 0,x^1,x^2)$

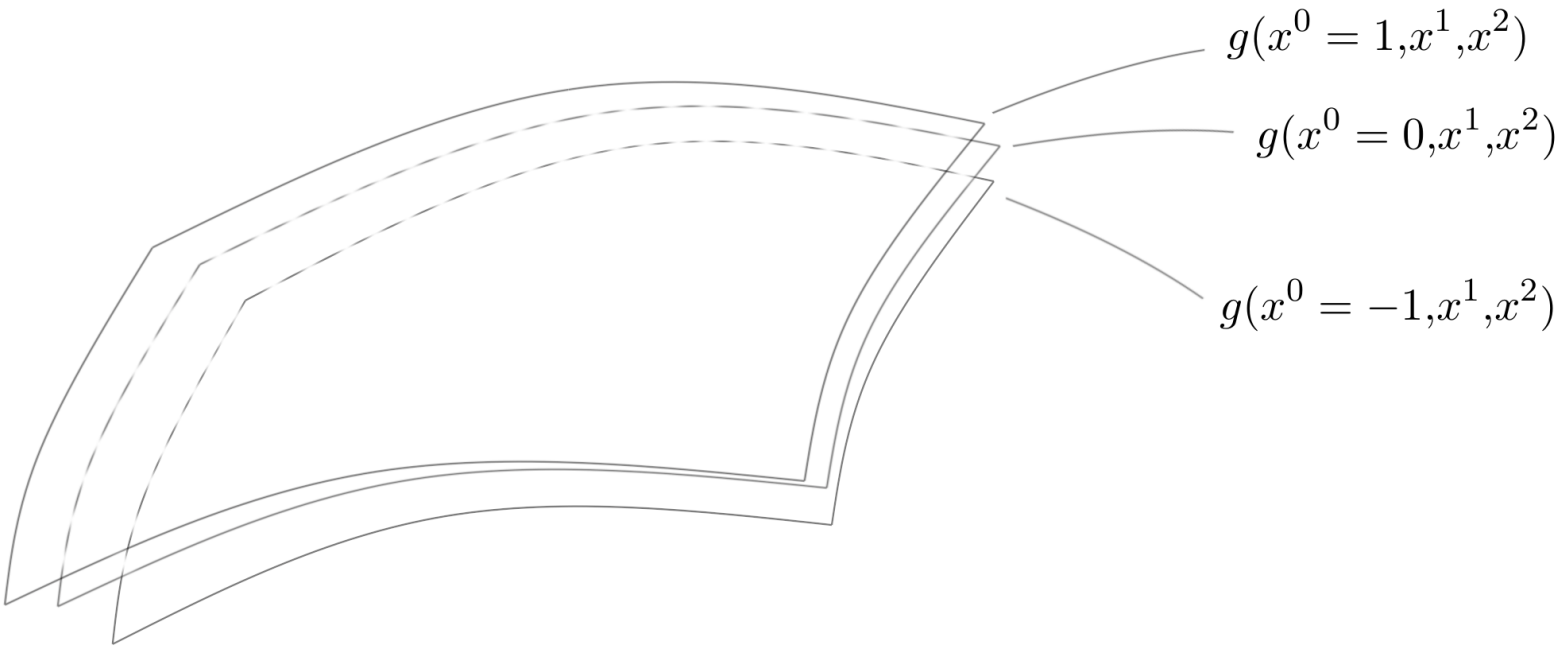
Manifold

Parallel manifolds:



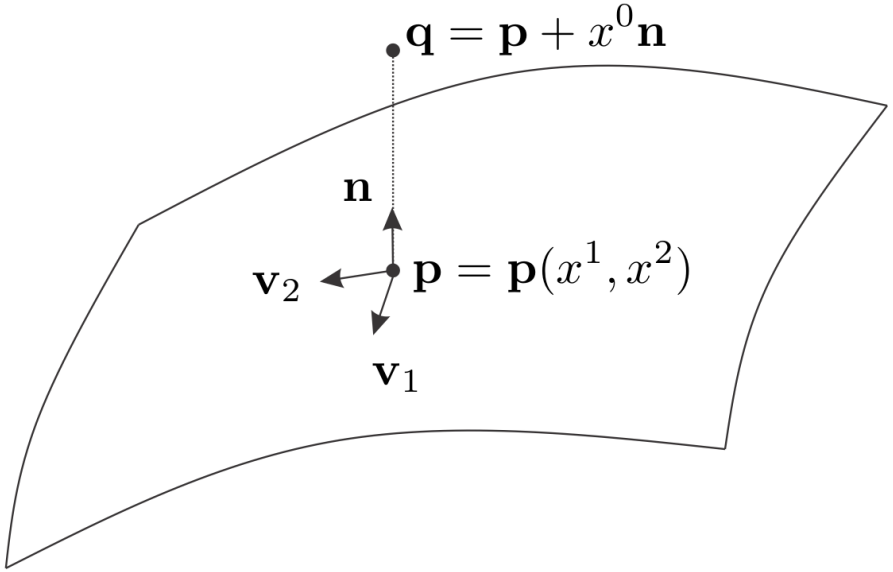
Manifold

Metric of parallel manifolds:



Manifold

Gaussian normal coordinate system:



Metric:

$$G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & & \\ 0 & & g(x^0,x^1,x^2) \end{pmatrix}$$

Metric determinant $\rightarrow$ Jacobian

$$dV = \sqrt{\det G}\, dx^0 dx^1 dx^2$$

Confinement potential: $V_{\text{harm}} = \dfrac{M\omega^2}{2} x^{0\,2}$

Energy, Action and Gross-Pitaevskii Equation

Energy:

$$E = \int dV \Psi^* \left(-\mu - \frac{\hbar^2}{2M} \nabla^2 + \frac{M\omega^2}{2} (x^0)^2 + \frac{1}{2} g_{\mathsf{int}} N |\Psi|^2 \right) \Psi$$

Action

$$S = \int dt \int dV \Psi^* \left(i\hbar\partial_t - \frac{\hbar^2}{2M} \nabla^2 + \frac{M\omega^2}{2} (x^0)^2 + \frac{1}{2} g_{\mathsf{int}} N |\Psi|^2 \right) \Psi$$

Energy, Action and Gross-Pitaevskii Equation

Extremize energy with respect to Ψ^*:

$$\left(-\mu - \frac{\hbar^2}{2M}\nabla^2 + \frac{M\omega^2}{2}(x^0)^2 + g_{\mathsf{int}}N|\Psi|^2\right)\Psi = 0$$

Extremize action with respect to Ψ^*:

$$\left(-i\hbar\partial_t - \frac{\hbar^2}{2M}\nabla^2 + \frac{M\omega^2}{2}(x^0)^2 + g_{\mathsf{int}}N|\Psi|^2\right)\Psi = 0$$

Notation: $\mathsf{GP}(\Psi) = 0$

Ansatz

Ansatz:

$\Psi(x^0,x^1,x^2) =$ depends on the variables x^0,x^1,x^2
and on the functions $\sigma(x^1,x^2),\ \psi(x^1,x^2)$

$\sigma(x^1,x^2) =$ cloud width;

$\psi(x^1,x^2) =$ two-dimensional wave function.

Ansatz

Possible choice

$$\Phi(x^0,x^1,x^2) = \frac{e^{-x^{0\,2}/2\sigma(x^1,x^2)^2}}{\sqrt[4]{\pi}\sqrt{\sigma(x^1,x^2)}} \cdot \phi(x^1,x^2)$$

Expected behaviour for an ansatz

$$\mathsf{GP}(\Psi) \simeq 0$$

This ansatz (for a non-flat manifold)

$$\mathsf{GP}(\Phi) \not\simeq 0$$

Inspired by L. Salasnich, A. Parola, L. Reatto, Phys. Rev. A **65**, 043614 (2002).

Ansatz

Previous choice

$$\Phi(x^0,x^1,x^2) = \frac{e^{-x^{0\,2}/2\sigma(x^1,x^2)^2}}{\sqrt[4]{\pi}\sqrt{\sigma(x^1,x^2)}} \cdot \phi(x^1,x^2)$$

New choice

$$\Psi(x^0,x^1,x^2) = \frac{e^{-x^{0\,2}/2\sigma(x^1,x^2)^2}}{\sqrt[4]{\pi}\sqrt{\sigma(x^1,x^2)}} \cdot \psi(x^1,x^2) \cdot \frac{\sqrt[4]{\det g(x^1,x^2)}}{\sqrt[4]{\det g(x^0,x^1,x^2)}}$$

For the last case we indeed have

$$\mathrm{GP}(\Psi) \simeq 0$$

Reducing Dimensionality

Energy:

$$E = \int dV \Psi^* \left(-\frac{\hbar^2}{2M} \nabla^2 + \frac{M\omega^2}{2} (x^0)^2 + \frac{1}{2} g_{\text{int}} N |\Psi|^2 \right) \Psi$$

Ansatz:

$$\Psi(x^0,x^1,x^2) = \frac{e^{-x^{0\,2}/2\sigma(x^1,x^2)^2}}{\sqrt[4]{\pi}\sqrt{\sigma(x^1,x^2)}} \cdot \psi(x^1,x^2) \cdot \frac{\sqrt[4]{\det g(x^1,x^2)}}{\sqrt[4]{\det g(x^0,x^1,x^2)}}$$

-Integrate approximately variable perpendicular to the manifold x^0.

-Extremize energy with respect to $\psi^*(x^1,x^2)$ and to $\sigma(x^1,x^2)$.

Inspired by L. Salasnich, A. Parola, L. Reatto, Phys. Rev. A **65**, 043614 (2002).

Extremizing the Energy

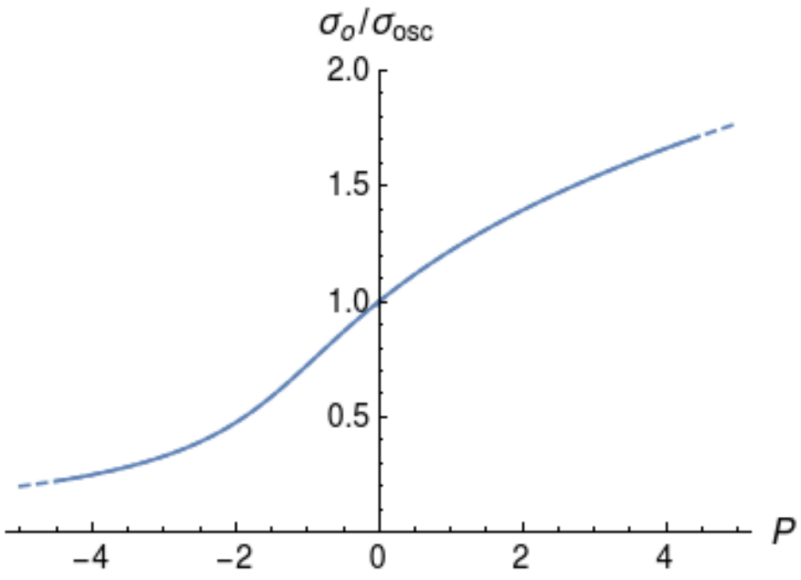
Extremizing the energy with respect to $\sigma(x^1,x^2)$:

Algebraic equation for σ:

$$\frac{\sigma^4}{\sigma_{\text{osc}}^4} = 1 + \frac{g_{\text{int}} M N}{\sqrt{2\pi}\hbar^2}\sigma|\psi|^2, \qquad \sigma_{\text{osc}} = \sqrt{\frac{\hbar}{M\omega}}$$

Extremizing the Energy on Sphere

Cloud width σ on a sphere of radius R:



$$P = \frac{a_s \sigma_{\rm osc} N}{\sqrt{2\pi}\ R^2}$$

$$a_s = 100 a_B, \sigma_{\rm osc} = 1\mu{\rm m}, R = 10\mu{\rm m}, N = 10^5 \Rightarrow P = 2$$

Extremizing the Energy

Extremize energy with respect to $\psi^*(x^1,x^2)$:

Two dimensional GP equation:

$$\mu\psi = \left(-\frac{\hbar^2}{2M}\nabla^2_{\mathrm{LB}} + V_{\mathsf{eff}}(x^1,x^2) + \frac{\hbar^2}{4M\sigma^2} + \frac{M\omega^2\sigma^2}{4} + \frac{g_{\mathsf{int}}N|\psi|^2}{\sqrt{2\pi}\sigma}\right)\psi$$

Extremizing the Energy

Extremize energy with respect to $\psi^*(x^1,x^2)$:

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$\nabla^2_{\mathrm{LB}} =$ Laplace-Beltrami operator

$V_{\mathsf{eff}}(x^1,x^2) =$ effective potential

∇^2_{LB} and $V_{\mathsf{eff}}(x^1,x^2)$ depend on the manifold.

Extremizing the Energy

Sphere of radius R:

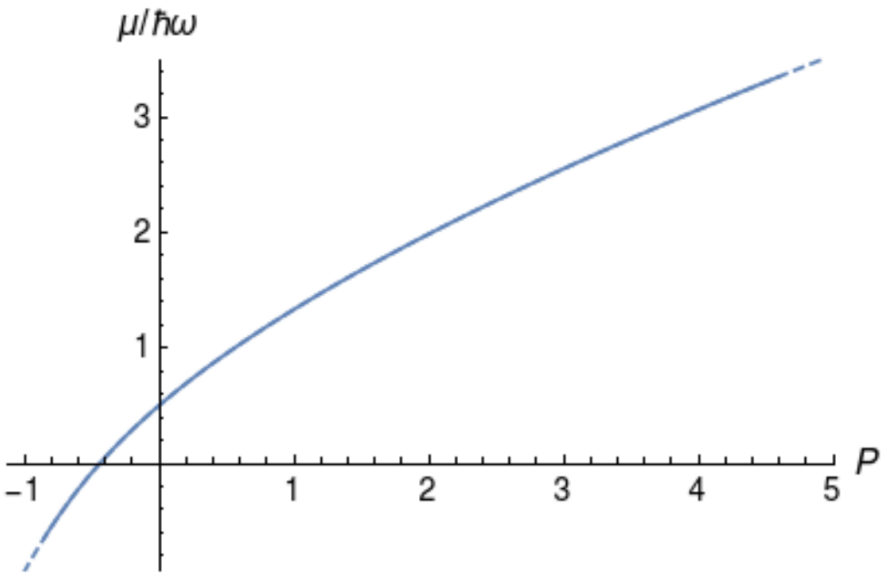
$$V_{\mathsf{eff}}(x^1,x^2) = 0$$

$$\nabla^2_{\mathsf{LB}} = -\frac{\mathbf{L}^2}{\hbar^2 R^2}$$

Ground state: $|\psi|^2 =$ uniform.

Extremizing the Energy

Chemical potential μ on a sphere of radius R:



Low-Lying Excitations on Quasi Sphere

Low-lying excitations:

-extremize the action, instead of the energy;

-perform a linear stability analysis.

Angular momentum operator:

$$\nabla^2_{\mathsf{LB}} = -\frac{\mathbf{L}^2}{\hbar^2 R^2}, \quad \text{eigenvalues of } \mathbf{L}^2: \quad \hbar^2 l(l+1)$$

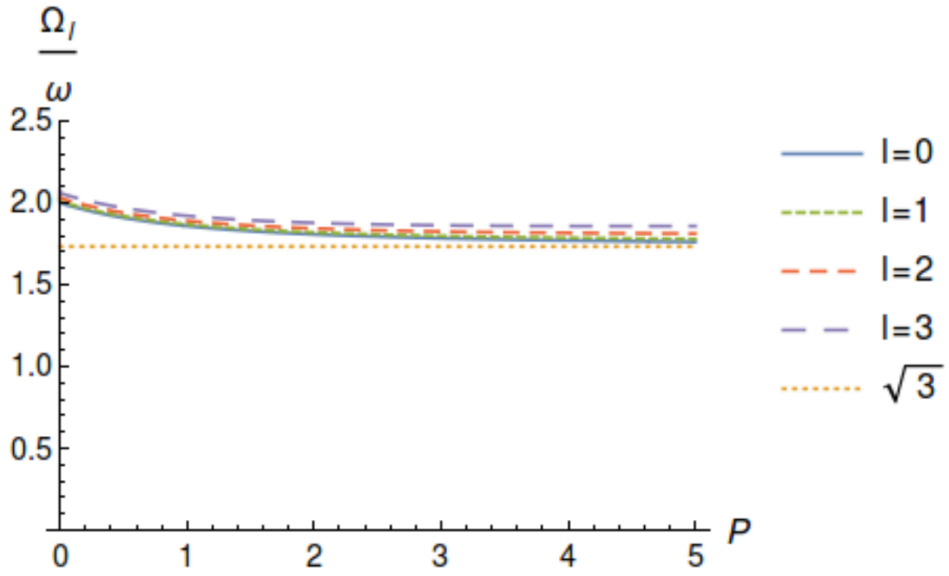
Low-Lying Excitations on Quasi Sphere

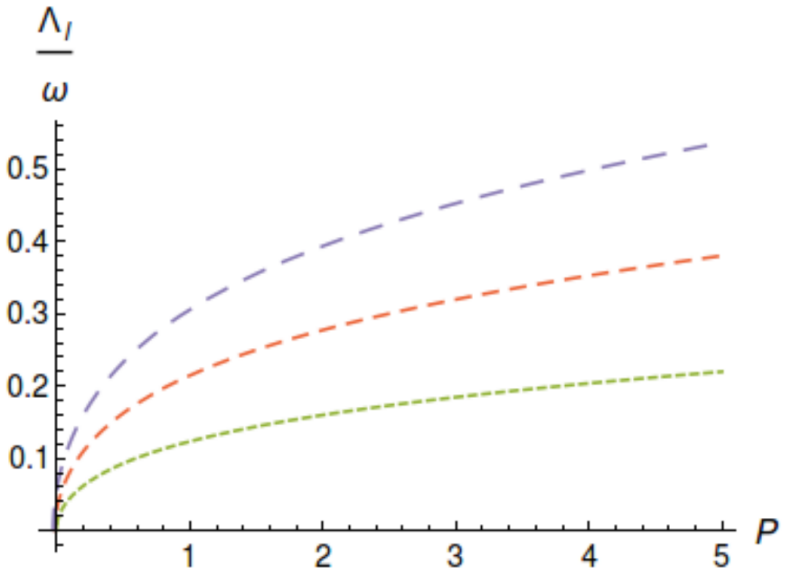
Higher frequencies:

$$\Omega_l = 2\omega + \frac{\hbar}{2MR^2} l(l+1) + \mathcal{O}(P)$$

Lower frequencies:

$$\Lambda_l = \frac{\hbar}{2MR^2} l(l+1) + \mathcal{O}(P)$$

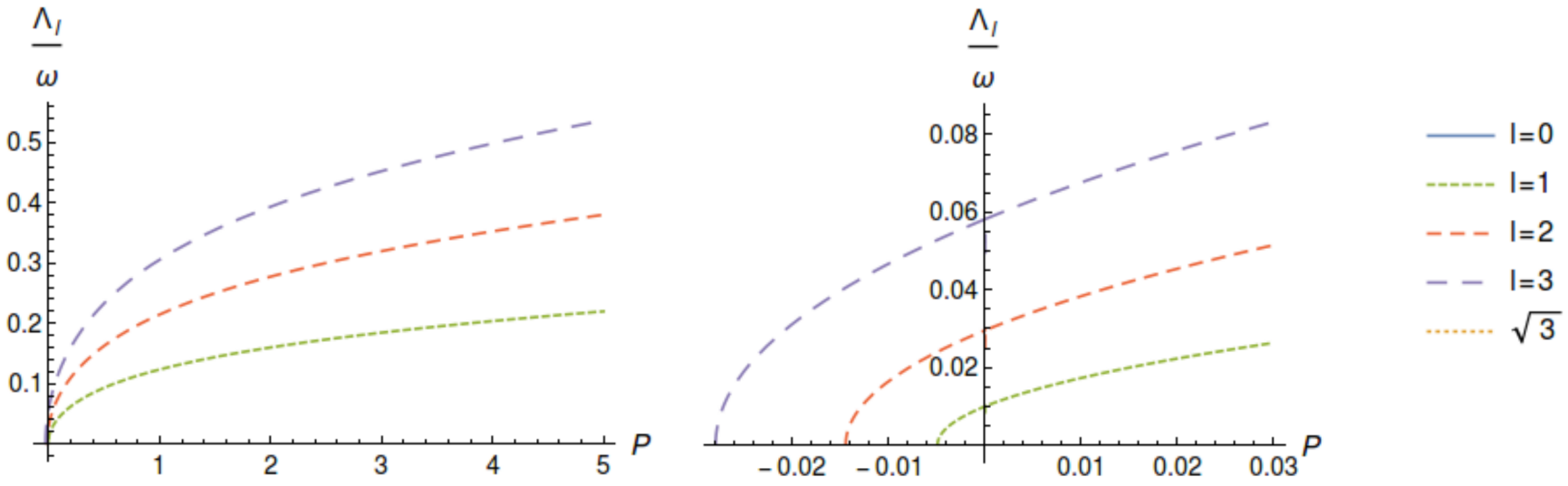




Low-Lying Excitations on Quasi Sphere

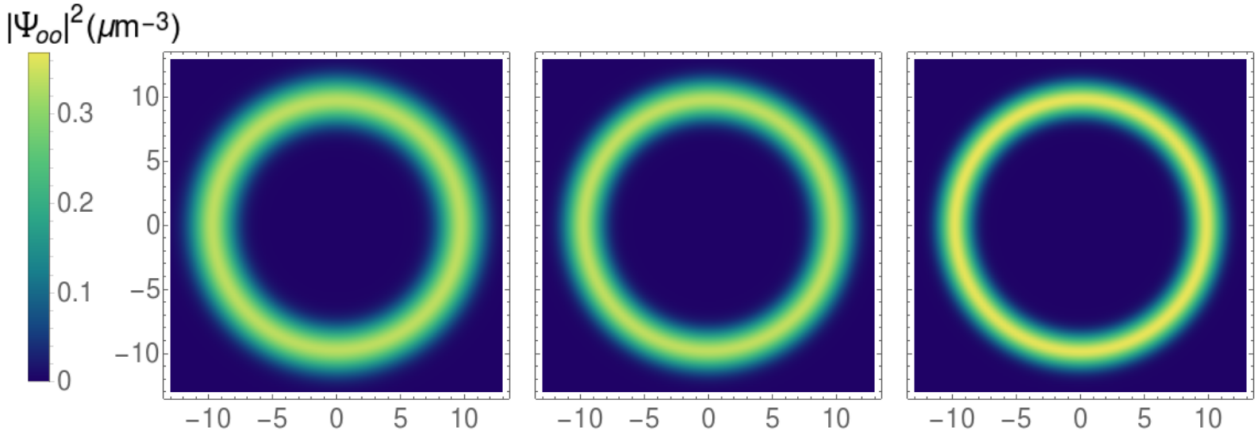
Lower frequencies: oscillation predominantly of the 2D wave function, *i.e.*, oscillations on the sphere.

$$\Lambda_l = \frac{\hbar}{2MR^2} l(l+1) + \mathcal{O}(P)$$



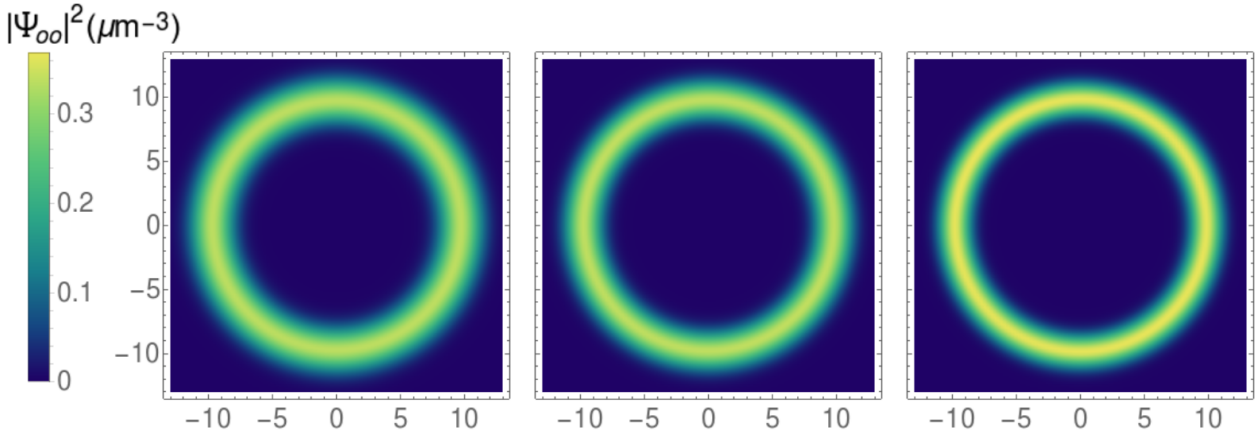
Modes

Accordion mode: quantum number $l = 0$



Modes

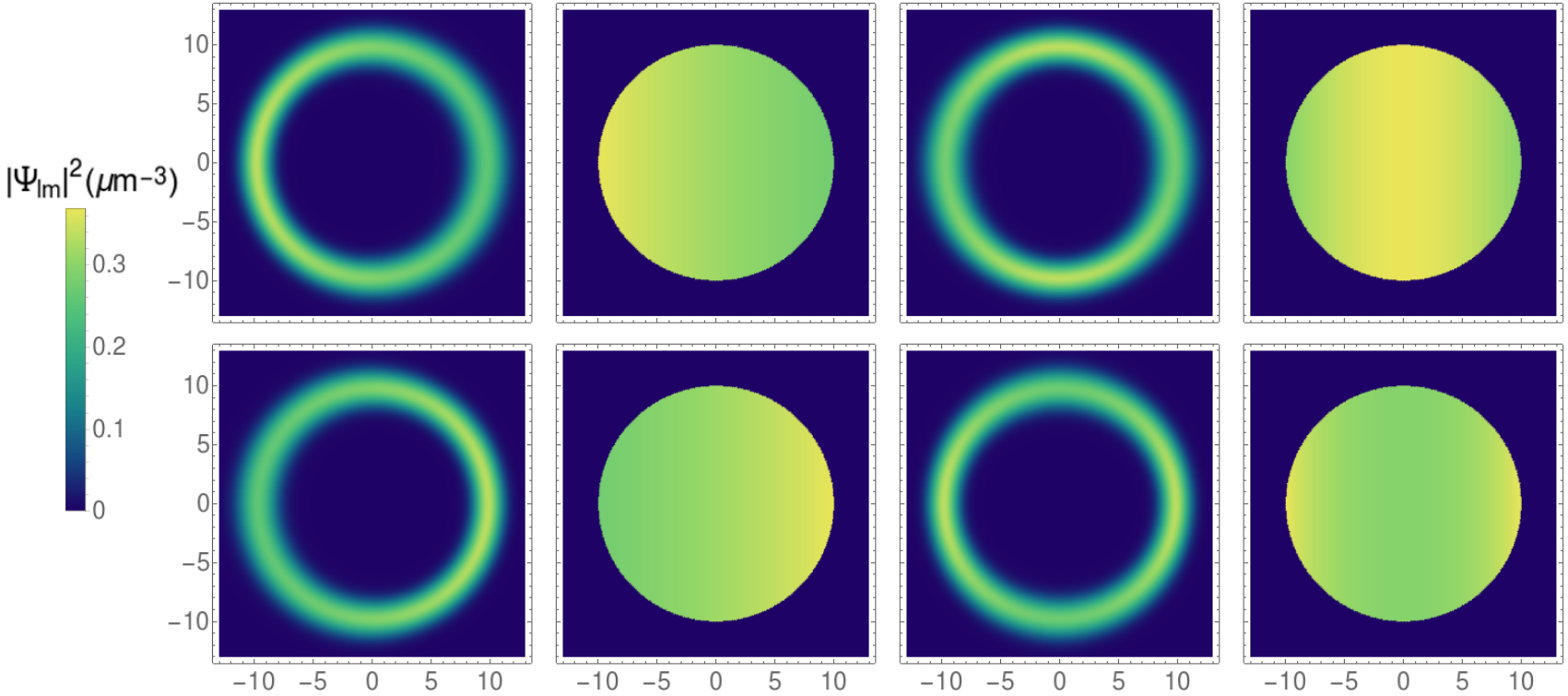
Accordion mode: quantum number $l = 0$



Breathing mode: sphere radius R would oscillate (here, R is fixed).

Modes

Quantum number $l = 1$ and 2



Sphere and More General Manifolds

Sphere:

-Equilibrium results

-Low-lying excitations modes

Sphere and More General Manifolds

Sphere:

-Equilibrium results

-Low-lying excitations modes

Other manifolds, ellipsoids...

More General Manifolds

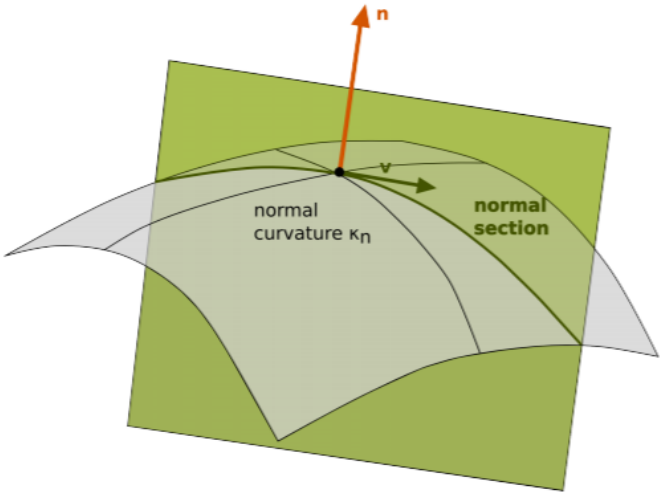
Effective potential: $V_{\text{eff}} \neq 0$

Figure from notes of Lia Vas

More General Manifolds

Effective potential: $V_{\text{eff}} \neq 0$

Sectional curve



Curvature of this curve: κ

Principal curvatures:

$\kappa_1 = \min \kappa, \quad \kappa_2 = \max \kappa$

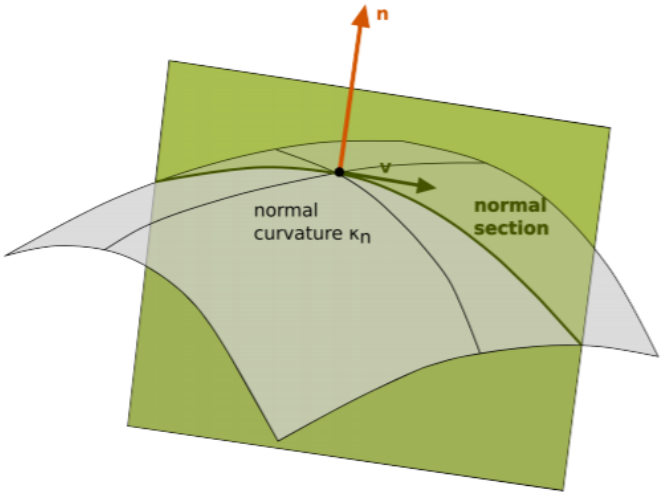
Gaussian curvature: $K = \kappa_1 \kappa_2$

Figure from notes of Lia Vas

More General Manifolds

Effective potential: $V_{\text{eff}} \neq 0$

Sectional curve



Curvature of this curve: κ

Principal curvatures:

$\kappa_1 = \min \kappa, \quad \kappa_2 = \max \kappa$

Gaussian curvature: $K = \kappa_1 \kappa_2$

Effective potential: $V_{\text{eff}} = -\dfrac{\hbar^2}{8M}(\kappa_1 - \kappa_2)^2$

Figure from notes of Lia Vas

Effective Potential in the Literature

Effective potential: $V_{\text{eff}} = -\frac{\hbar^2}{8M}(\kappa_1 - \kappa_2)^2$

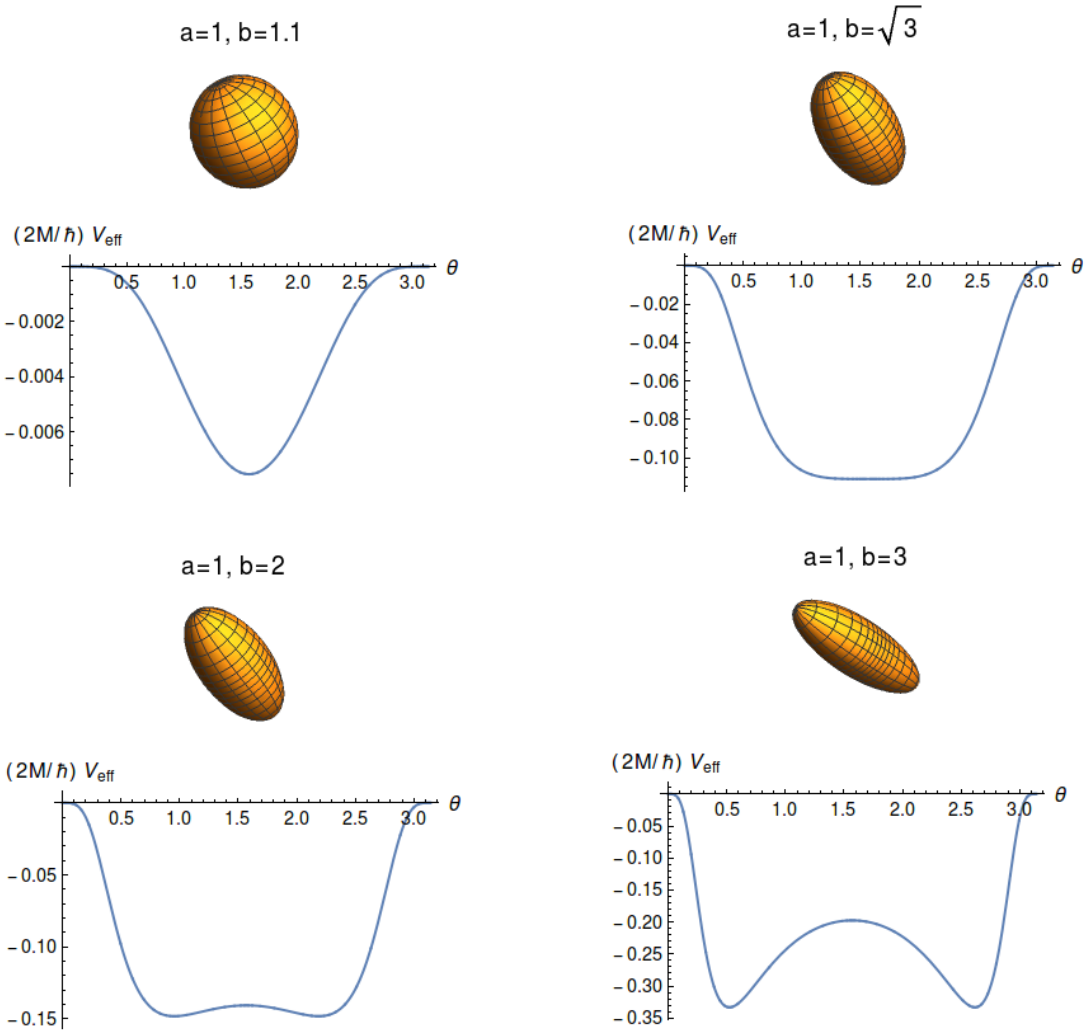
R. C. T. da Costa, Phys. Rev. A **23**, 1982 (1981).

L. Kaplan, N. T. Maitra, and E. J. Heller, Phys. Rev. A **56**, 2592 (1997).

P. Sandin, M. Ogren, M. Gulliksson, J. Smyrnakis, M. Magiropoulos, and G. M. Kavoulakis, Phys. Rev. E **95**, 012142 (2017).

Effective Potential on Ellipsoids

Prolate ellipsoids



Two-dimensional GP equation:

Two-dimensional GP equation, considering $\sigma(x^1,x^2) = \sigma_{\mathsf{osc}}(x^1,x^2)$:

$$\mu\psi = \left(-\frac{\hbar^2}{2M}\nabla^2_{\mathrm{LB}} + V_{\mathsf{eff}}(x^1,x^2) + \frac{\hbar^2}{2M\sigma^2(x^1,x^2)} + \frac{g_{\mathsf{int}}N|\psi|^2}{\sqrt{2\pi}\sigma(x^1,x^2)}\right)\psi$$

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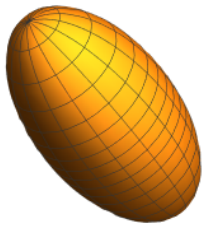
Two-dimensional GP equation, considering $\sigma(x^1,x^2) = \sigma_{\text{osc}}(x^1,x^2)$:

$$\mu\psi = \left(-\frac{\hbar^2}{2M}\nabla^2_{\text{LB}} + V_{\text{eff}}(x^1,x^2) + \frac{\hbar^2}{2M\sigma^2(x^1,x^2)} + \frac{g_{\text{int}}N|\psi|^2}{\sqrt{2\pi}\sigma(x^1,x^2)}\right)\psi$$

Ellipsoid:

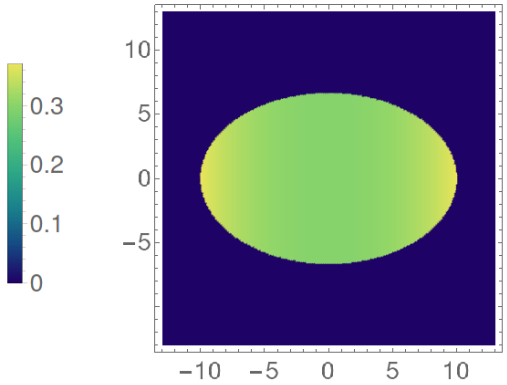


Uniform σ: higher density on equator;

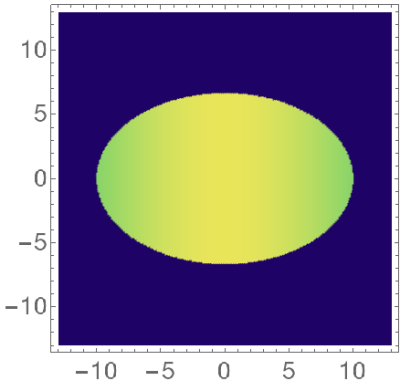
Non-uniform $\sigma = \sigma(\theta,\varphi)$: higher density on the poles (bubble trap case).

Ground State on an Ellipsoid

Non-uniform
confinement frequency



Uniform
confinement frequency



N. S. Móller, F. E. A. dos Santos, and A. Pelster, in preparation.

Ground State on an Ellipsoid

Non-uniform
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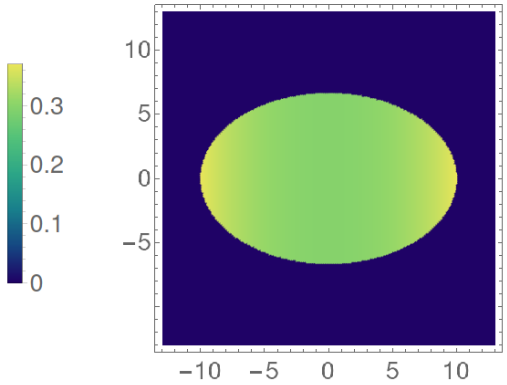
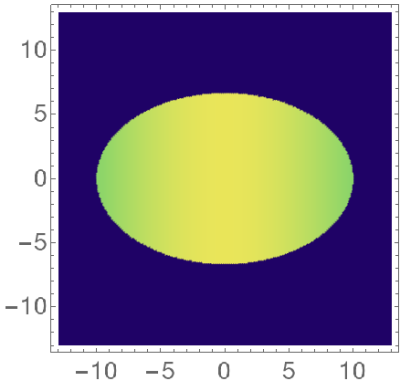


Behaviour due to
external asymmetry

Uniform
confinement frequency



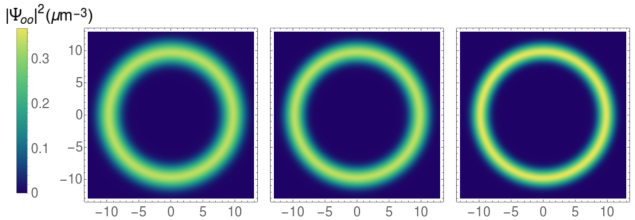
Behaviour due to
geometrical properties

N. S. Móller, F. E. A. dos Santos, and A. Pelster, in preparation.

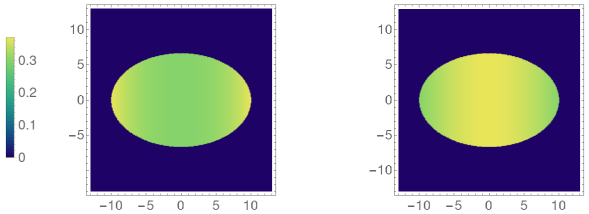
Conclusions

-Quasi two-dimensional equations for the wave function of a gas confined on a curved manifold.

-Equilibrium and low-lying excitations on quasi sphere:



-Effective potentials:



Thank you!

Ansatz and Normalization

The new ansatz provides a better approximation to the particle number

$$\Phi(x^0,x^1,x^2):$$

$$N = \int dx^1 dx^2 \sqrt{\det g(x^1,x^2)} |\phi(x^1,x^2)|^2 + \mathcal{O}\left(\frac{\sigma^2}{R^2}\right).$$

$$\Psi(x^0,x^1,x^2):$$

$$N = \int dx^1 dx^2 \sqrt{\det g(x^1,x^2)} |\psi(x^1,x^2)|^2 + \mathcal{O}(e^{-R^2/\sigma^2})$$

Low-Lying Excitations on Quasi Sphere

Higher frequencies: oscillation predominantly of the cloud thickness.

$$\Omega_l = 2\omega + \frac{\hbar}{2MR^2} l(l+1) + \mathcal{O}(P)$$