Ginzburg-Landau Theory for the Jaynes-Cummings Hubbard Model

Christian Nietner

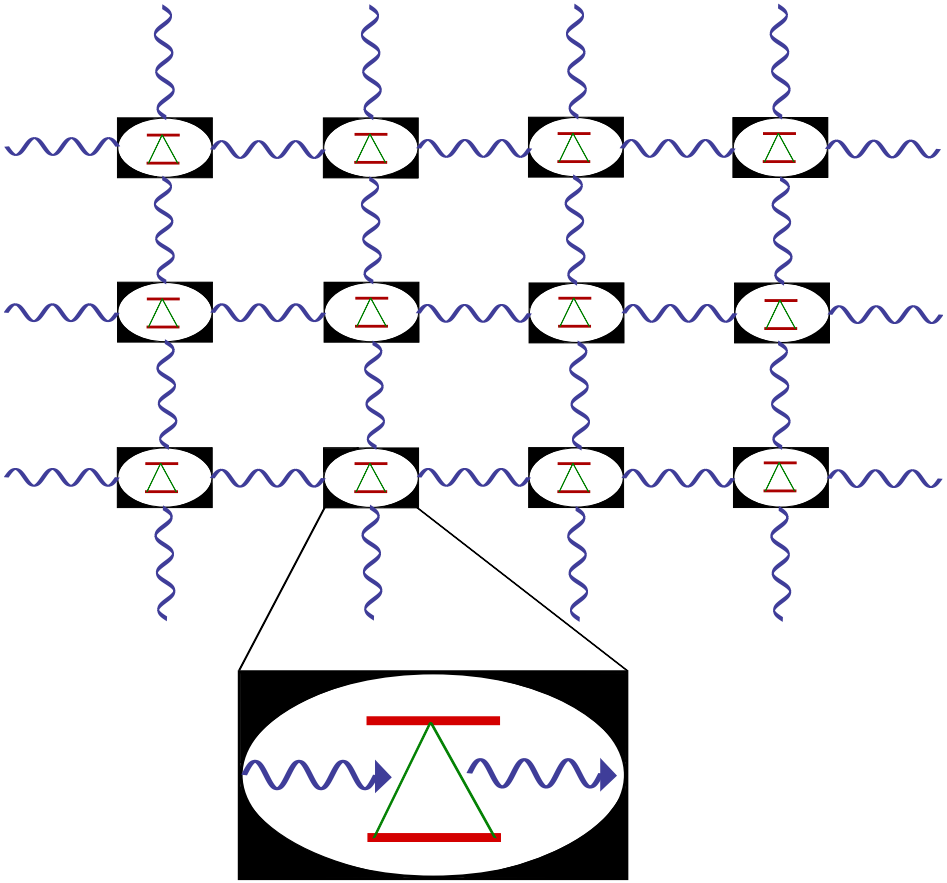
Institut für Theoretische Physik, Freie Universität Berlin, Germany

Seminar Talk 22.10.2010

Outline

Jaynes-Cummings-Hubbard Model

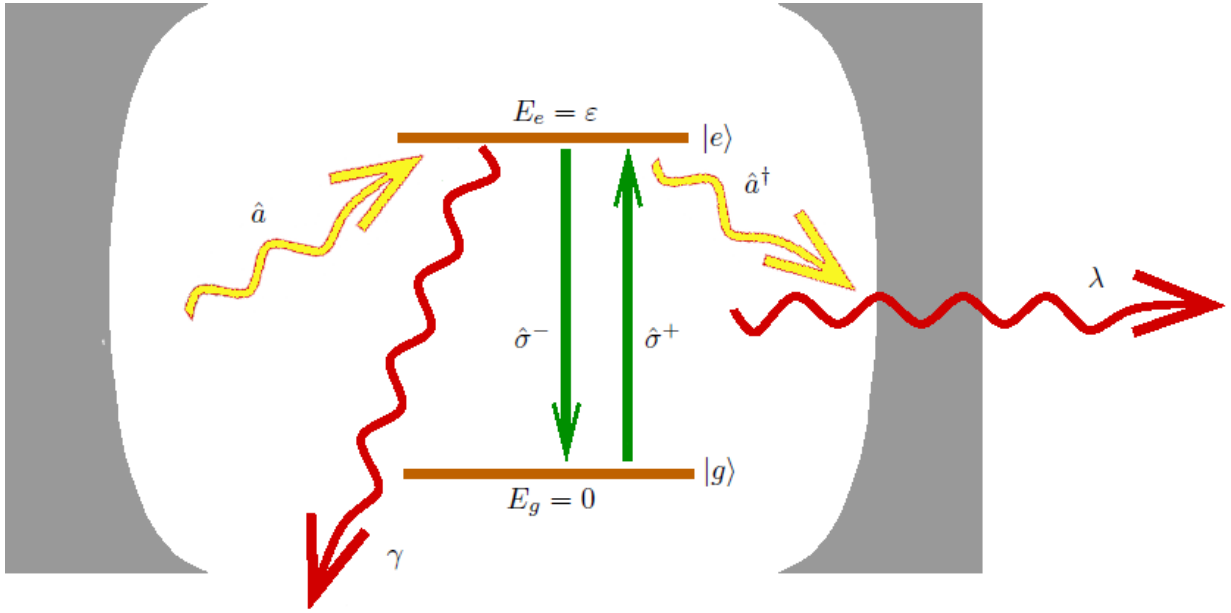
- model for strongly correlated quantum systems
- periodical structure built of micro cavities
- each cavity contains a two-level system
- photon hopping between next neighbours
- exhibits Mott-insulator and superfluid phase



M. J. Hartmann, F. G. S. L. Brandão, M. B. Plenio, Laser & Photon. Rev. 2, **6**, 527 (2008)

The Jaynes-Cummings Model

- suggested in 1963; a cornerstone of quantum optics



- describes the interaction of a two level system with a monochromatic electromagnetic field in RWA

Hamiltonian:

$$\hat{\mathcal{H}}^{\mathrm{JC}} = \omega\, \hat{a}^\dagger \hat{a} + \varepsilon\, \hat{\sigma}^+ \hat{\sigma}^- + g\left(\hat{a}\, \hat{\sigma}^+ + \hat{a}^\dagger \hat{\sigma}^-\right)$$

The Jaynes-Cummings Hamiltonian

- Hamiltonian in a more convenient form

Hamiltonian:

$$\hat{\mathcal{H}}^{\mathrm{JC}} = \omega\,\hat{N} + \Delta\,\hat{\sigma}^{+}\hat{\sigma}^{-} + g\left(\hat{a}\,\hat{\sigma}^{+} + \hat{a}^{\dagger}\hat{\sigma}^{-}\right)$$

- with polariton occupation number operator

occupation number operator:

$$\hat{N} = \hat{a}^{\dagger}\hat{a} + \hat{\sigma}^{+}\hat{\sigma}^{-}$$

- and the detuning parameter

detuning:

$$\Delta = \varepsilon - \omega$$

Polariton states

- $[\hat{\mathcal{H}}^{\mathrm{JC}}, \hat{N}] = 0$, hence the conserved quantity in this model is the polariton number
- polaritons: coupled excitations of the atom and the field cavity mode

generation of polariton state:

$$|\psi_{\mathrm{pol}}\rangle = |\psi_{\mathrm{field}}\rangle \otimes |\psi_{\mathrm{atom}}\rangle = |n\rangle \otimes \begin{pmatrix} |g\rangle \\ |e\rangle \end{pmatrix}$$

- note: for a fixed number n of polaritons there exist two possible micro states

n-polariton state

$$|\psi_n\rangle = |n, g\rangle + |n-1, e\rangle$$

Jaynes-Cummings eigenstates

- solving the Jaynes-Cummings Hamiltonian in polariton basis yields

energy eigenvalues:

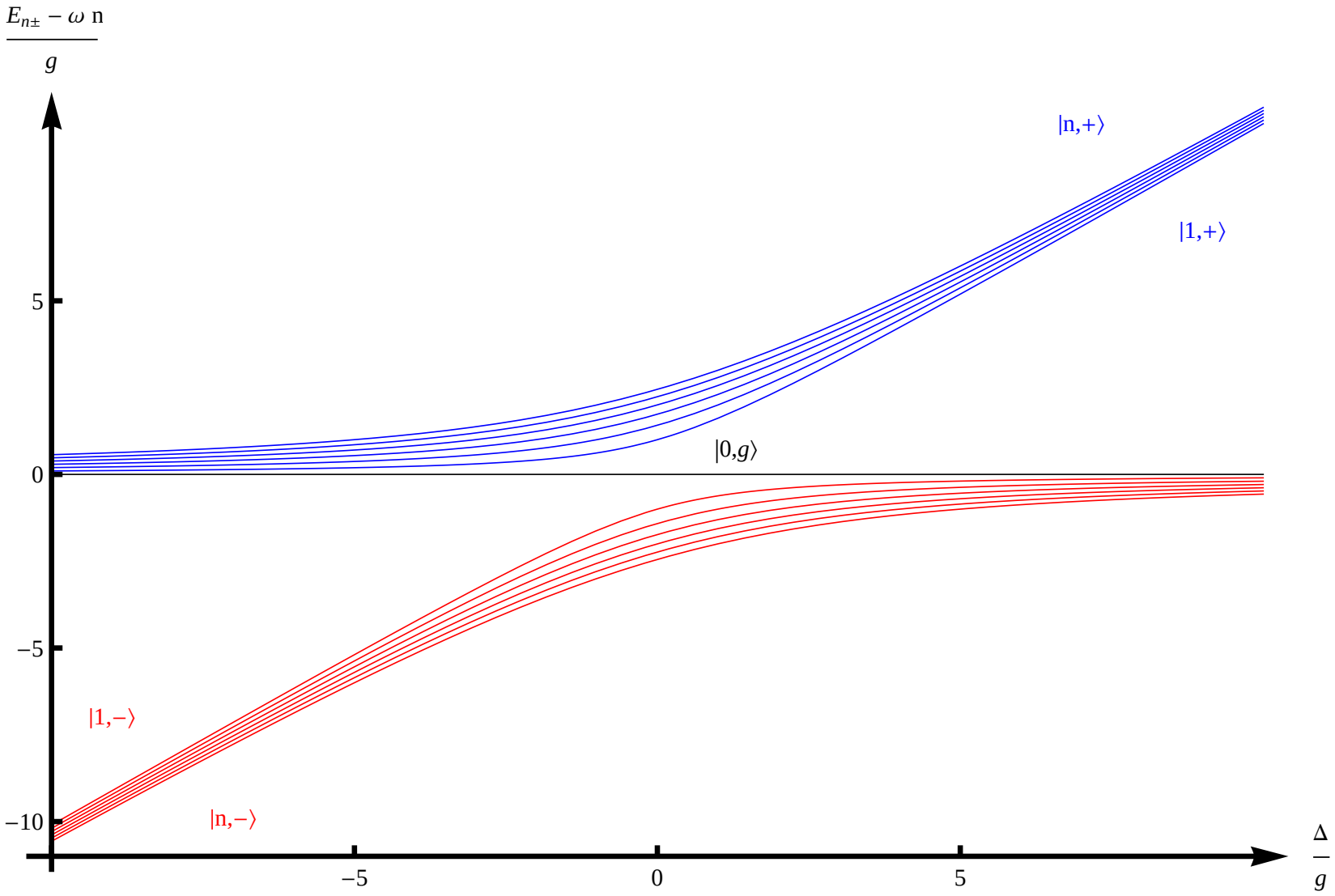
$$E_{n\pm} = \omega n + \frac{1}{2}\left(\Delta \pm \sqrt{\Delta^2 + 4g^2 n}\right), \quad (n > 1), \quad E_0 = 0$$

eigenstates:

$$|n,+\rangle = \sin\theta_n |n,g\rangle + \cos\theta_n |n-1,e\rangle$$
$$|n,-\rangle = \cos\theta_n |n,g\rangle - \sin\theta_n |n-1,e\rangle$$

- with mixing angle: $\theta_n = \frac{1}{2}\arctan\left(\frac{2g\sqrt{n}}{\Delta}\right)$

Polariton branches



Jaynes-Cummings Hubbard Hamiltonian

- describes lattice of coupled Jaynes-Cummings systems
- introducing hopping term due to wave-function overlap with hopping probability κ_{ij}
- furthermore we are working in the grand canonical ensemble and therefore we get an extra term $\mu\hat{N}$

Jaynes-Cummings Hubbard Hamiltonian:

$$\hat{\mathcal{H}} = -\sum_i \mu_{\text{eff}}\,\hat{N}_i + \Delta\,\hat{\sigma}_i^+\hat{\sigma}_i^- + g\left(\hat{a}_i\,\hat{\sigma}_i^+ + \hat{a}_i^\dagger\hat{\sigma}_i^-\right) - \sum_{ij}\kappa_{ij}\,\hat{a}_i^\dagger\hat{a}_j$$

- with effective chemical potential $\mu_{\text{eff}} = \mu - \omega$

Eigenvalues of unperturbed Hamiltonian

- Hamiltonian decomposes into an unperturbed and analytically solvable part and a perturbation part

splitting of the Hamiltonian

$$\hat{\mathcal{H}}_0 = \sum_i \hat{\mathcal{H}}_i^{\mathrm{JC}} - \mu \hat{N}\,, \qquad \hat{\mathcal{H}}_1 = -\sum_{ij} \kappa_{ij}\, \hat{a}_i^{\dagger}\, \hat{a}_j$$

- energy eigenvalues of unperturbed Hamiltonian

energy eigenvalues of unperturbed JCHM:

$$E_{n\pm} = -\mu_{\mathrm{eff}}\, n + \frac{1}{2}\left(\Delta \pm \sqrt{\Delta^2 + 4g^2 n}\right)$$

- perturbation part corresponds to hopping

Grand canonical free energy

- phase boundary and thermodynamic response functions from grand canonical free energy
- calculate partition function in Dirac Interaction Picture

$$\mathcal{Z} = \mathrm{Tr}\left\{e^{-\beta\hat{\mathcal{H}}_0}\hat{U}_\mathrm{D}(\beta,0)\right\}, \quad \hat{U}_\mathrm{D}(\tau,\tau_0) = \hat{T}e^{-\int_{\tau_0}^{\tau} d\tau \hat{\mathcal{H}}_{1\mathrm{D}}(\tau)}$$

Partition function:

$$\mathcal{Z} = \mathcal{Z}_0\left\langle \hat{U}_\mathrm{D}(\beta,0)\right\rangle_0, \quad \langle\bullet\rangle_0 = \frac{1}{\mathcal{Z}_0}\mathrm{tr}\left\{\bullet\, e^{-\beta\hat{\mathcal{H}}_0}\right\}$$

- expanding the exponential yields

$$\mathcal{Z} = \mathcal{Z}_0\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_0^\beta d\tau_1\ldots\int_0^\beta d\tau_n\left\langle \hat{T}\left[\hat{\mathcal{H}}_{1\mathrm{D}}(\tau_1)\ldots\hat{\mathcal{H}}_{1\mathrm{D}}(\tau_n)\right]\right\rangle_0$$

averages correspond to n-particle Green's functions

How to proceed?

- modify Jaynes-Cummings-Hubbard Hamiltonian to:

$$\hat{\mathcal{H}}' \to \hat{\mathcal{H}} + \Delta\hat{\mathcal{H}}$$

- introduce symmetry breaking currents $j(\tau)$, $j^*(\tau)$ coupling to $\hat{a}$, $\hat{a}^\dagger$ making $\mathcal{F}$, $\mathcal{Z}$ functionals of the currents:

$$\Delta\hat{\mathcal{H}} = \sum_i \left(j_i(\tau)\, \hat{a}_i^\dagger + j_i^*(\tau)\, \hat{a}_i \right)$$

- physical results are consistent if calculations are evaluated at $j_i(\tau) = j_i^*(\tau) = 0$ in the end
- advantage: allows for diagrammatic *linked cluster expansion* of the grand canonical free energy

see B.Bradlyn, F.E.A. dos Santos, A.Pelster; Phys. Rev. A 79, 013615 (2009)

Cluster expansion

- expansion of $\mathcal{Z}$ yields sum of n-particle Green's functions
- linked cluster expansion: Green's functions decompose into a sum of products of cumulants

generating cumulant

$$C_0^{(0)}[j,j^*] = \ln\left(\frac{\mathcal{Z}[j,j^*]}{\mathcal{Z}_0}\right)$$

- get higher cumulants as functional derivatives

higher cumulants

$$C_n^{(0)}\left(i_1',\tau_1'\ldots|i_1,\tau_1\ldots\right) = \left.\frac{\delta^{2n}C_0^{(0)}[j,j^*]}{\delta j_{i_1'}(\tau_1')\ldots\delta j_{i_n'}(\tau_n')\,\delta j_{i_1}^*(\tau_1)\ldots\delta j_{i_n}^*(\tau_n)}\right|_{j,j^*=0}$$

see W. Metzner; Phys. Rev. B 43, 8549 (1991)

Diagrammatic expansion

Diagrammatic rules

- 2n-th order cumulant corresponds to a vertex with n lines entering and n lines leaving
- Draw all topologically inequivalent connected diagrams
- Label each vertex with a site index, and each line with an imaginary-time variable
- internal lines represent a factor of κ_{ij}
- external entering (leaving) lines correspond to a factor $j_i(\tau)\left(j_i^*(\tau)\right)$
- multiply by the multiplicity and divide by the symmetry factor
- integrate over all internal time variables

Diagrammatic expansion of the free energy

- using the diagrammatic rules we find up to 4th order int currents and 1st order in hopping

- corresponding to the analytic expansion

$$
\begin{aligned}
\mathcal{F} =& F_0 - \frac{1}{\beta}\sum_i \Bigg\{ \int_0^\beta \mathrm{d}\tau_1 \int_0^\beta \mathrm{d}\tau_2 \left[a_2^{(0)}(i,\tau_1|i,\tau_2) j_i(\tau_1) j_i^*(\tau_2) + \sum_j a_2^{(1)}(i,\tau_1|j,\tau_2) t_{ij} j_i(\tau_1) j_j^*(\tau_2) \right] \\
&+ \frac{1}{4}\int_0^\beta \mathrm{d}\tau_1 \int_0^\beta \mathrm{d}\tau_2 \int_0^\beta \mathrm{d}\tau_3 \int_0^\beta \mathrm{d}\tau_4\, a_4^{(0)}(i,\tau_1;i,\tau_2|i,\tau_3;i,\tau_4) j_i(\tau_1) j_i(\tau_2) j_i^*(\tau_3) j_i^*(\tau_4) \\
&+ \frac{1}{2}\int_0^\beta \mathrm{d}\tau_1 \int_0^\beta \mathrm{d}\tau_2 \int_0^\beta \mathrm{d}\tau_3 \int_0^\beta \mathrm{d}\tau_4 \sum_j t_{ij} \Big[a_4^{(1)}(i,\tau_1;i,\tau_2|j,\tau_3;i,\tau_4) j_i(\tau_1) j_i(\tau_2) j_j^*(\tau_3) j_i^*(\tau_4) \\
&+ a_4^{(1)}(i,\tau_1;j,\tau_2|i,\tau_3;i,\tau_4) j_i(\tau_1) j_j(\tau_2) j_i^*(\tau_3) j_i^*(\tau_4) \Big] \Bigg\}
\end{aligned}
$$

Ginzburg-Landau effective action

- performing Matsubara transformation: $\omega_m = \frac{2\pi m}{\beta}$
- introduce Ginzburg-Landau order parameter

$$\Psi_i(\omega_m) = \langle \hat{a}_i(\omega_m) \rangle = \beta \frac{\delta \mathcal{F}}{\delta j_i^*(\omega_m)}$$

- perform Legendre transformation to the effective action Γ

Effective action

$$\Gamma[\Psi_i(\omega_m), \Psi_i^*(\omega_m)] = \mathcal{F} - \frac{1}{\beta} \sum_{i,\omega_m} [\Psi_i(\omega_m) j_i^*(\omega_m) + \Psi_i^*(\omega_m) j_i(\omega_m)]$$

- conjugate fields: $j_i(\omega_m) = -\beta \frac{\delta \Gamma}{\delta \psi_i^*(\omega_m)}$
- physical situation $j = 0$ becomes: $\frac{\delta \Gamma}{\delta \psi_i^*(\omega_m)} = 0$

B. Bradlyn, F. E. A. dos Santos, A. Pelster, Phys. Rev. A **79**, 013615 (2009)

Γ Expansion

- cluster expansion to 2. order in j and 1. order in κ

$$\Gamma\left[\psi_i\left(\omega_m\right), \psi_i^*\left(\omega_m\right)\right] \approx \mathcal{F}_0 + \frac{1}{\beta}\sum_{\omega_m}\left[\sum_{i,j}\frac{\delta_{ij}}{a_2^{(0)}(\omega_m)} - \kappa\sum_{\langle i,j\rangle}\right]\psi_i\left(\omega_m\right)\psi_j^*\left(\omega_m\right)$$

- physical situation for static field $\psi_i\left(\omega_m\right) = \sqrt{\beta}\,\psi\,\delta_{m,0}$ gets

$$0 \stackrel{!}{=} \left(\left[a_2^{(0)}\left(0\right)\right]^{-1} - \kappa\, z\right)\psi$$

$$a_2^{(0)}(0) = \frac{1}{\mathcal{Z}_0}\sum_{\alpha,\alpha'=\pm}\left\{\frac{(t_{1\alpha'\alpha})^2}{E_{1\alpha'}} - \sum_{n=1}^{\infty} e^{-\beta E_{n\alpha}}\left[\frac{\left(t_{(n+1)\alpha'\alpha}\right)^2}{E_{n\alpha}-E_{(n+1)\alpha'}} + \frac{(t_{n\alpha\alpha'})^2}{E_{n\alpha}-E_{(n-1)\alpha'}}\right]\right\}$$

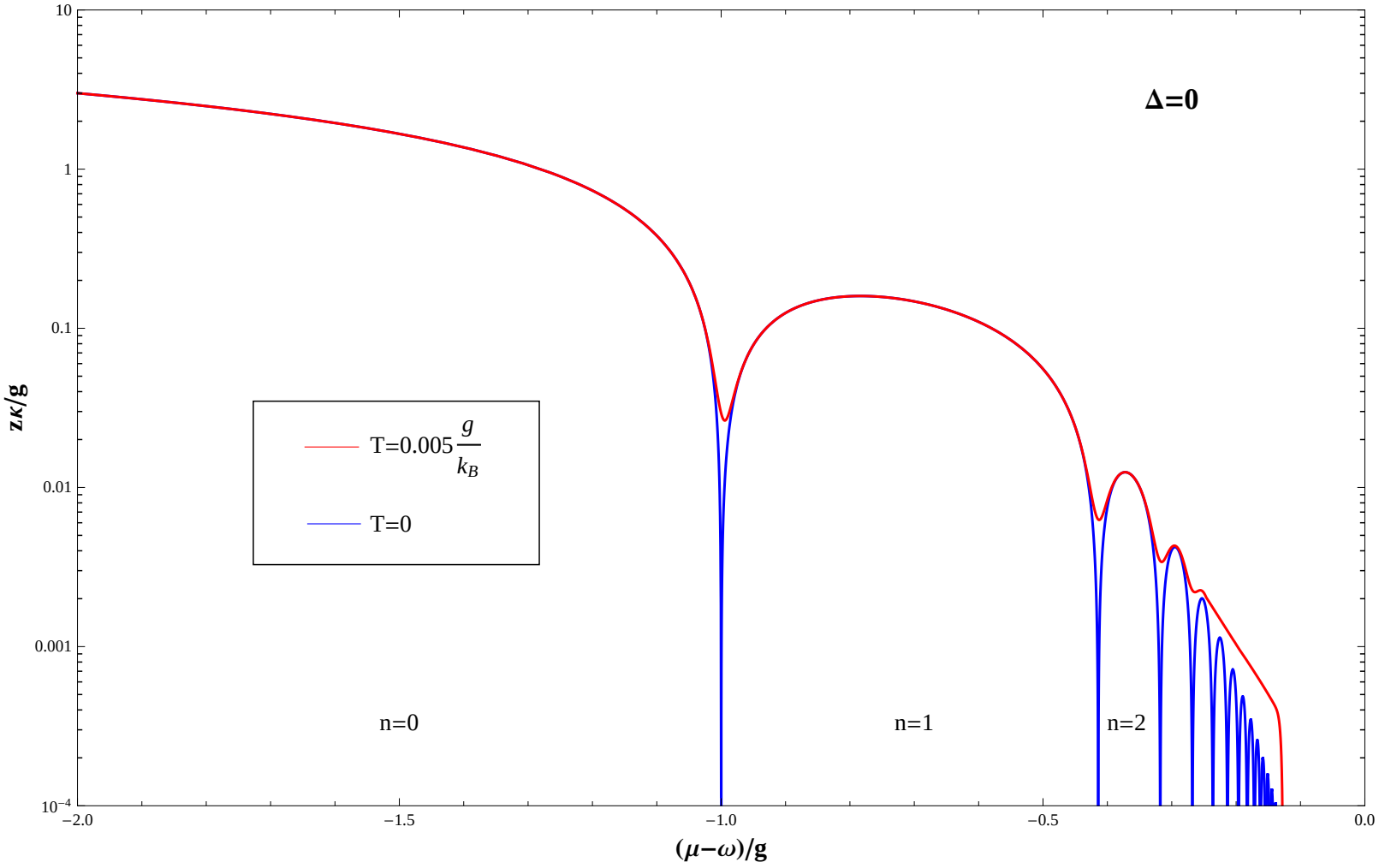
$$E_{n\alpha} = (\omega-\mu)\,n + \frac{1}{2}\left(\Delta + \alpha\sqrt{\Delta^2 + 4\,g^2\,n}\right),$$

$$\begin{aligned} t_{n\pm-} &= \sqrt{n}\,a_{n\pm}\,b_{n-1+} + \sqrt{n-1}\,b_{n\pm}\,b_{n-1-} \\ t_{n\pm+} &= \sqrt{n}\,a_{n\pm}\,a_{n-1+} + \sqrt{n-1}\,b_{n\pm}\,a_{n-1-} \end{aligned},$$

$$\begin{pmatrix} a_{n+} & b_{n+} \\ a_{n-} & b_{n-} \end{pmatrix} = \begin{pmatrix} \sin\theta_n & \cos\theta_n \\ \cos\theta_n & -\sin\theta_n \end{pmatrix}$$

$$\theta_n = \frac{1}{2}\arctan\left(\frac{2\,g\sqrt{n}}{\Delta}\right)$$

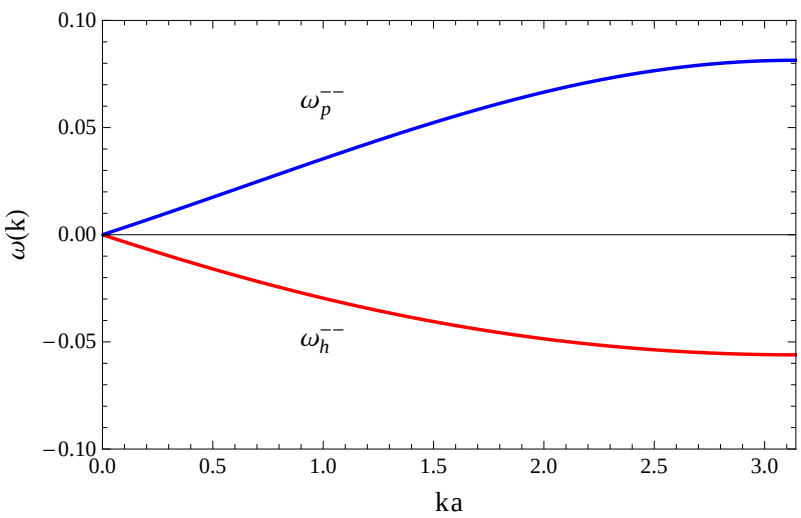
Phase boundary

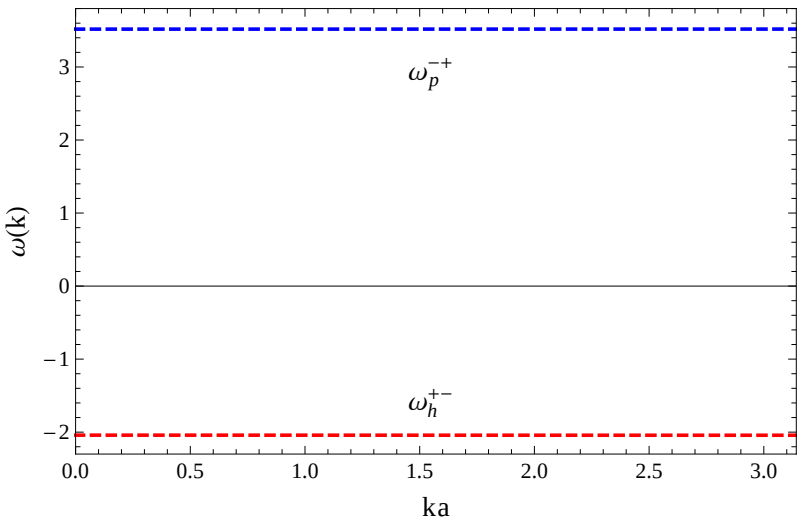


J. Koch, K. L. Hur, Phys. Rev. A, **80**, 023811 (2009)

Excitation Spectra in Mott Phase

- stem from divergence of correlation function



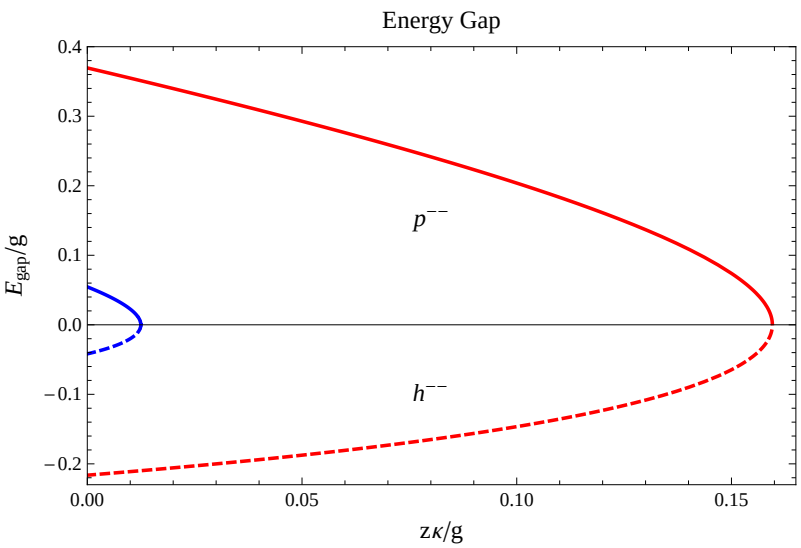


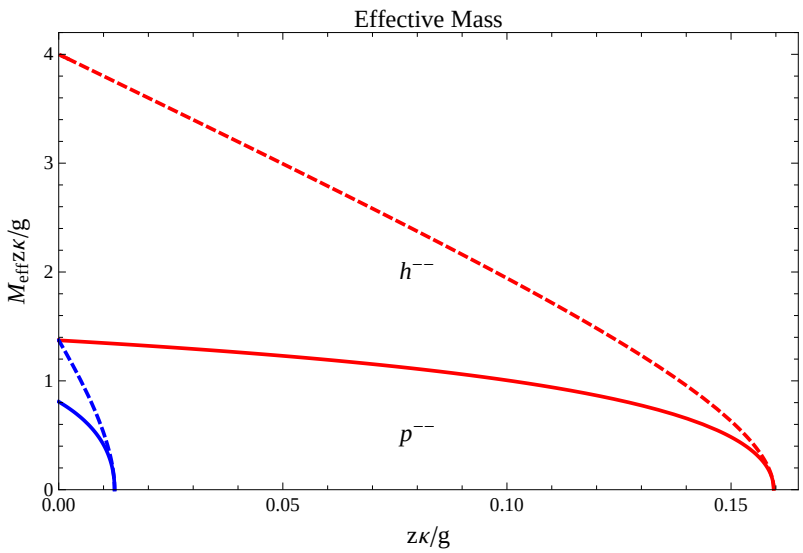
above graphics for: $\Delta = 0,\ T = 0,\ n = 2$ (tip of lobe)

Energy Gap and Effective Mass in Mott Phase

- quantum phase transition determined by lower polariton branch

$$(\Delta = 0,\ T = 0)$$





red graphs for: $n = 1$ (tip of lobe) blue graphs for: $n = 2$ (tip of lobe)

S. Schmidt, G. Blatter, Phys. Rev. Lett. **103**, 086403 (2009)

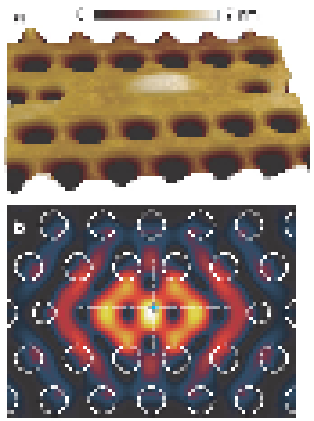
Experimental realizations

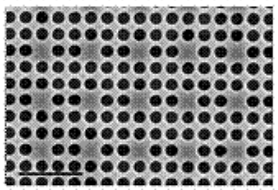
- implementation of Jaynes-Cummings system since decades
- started with relatively huge Fabry-Perot cavities
- today already routine production of arrays of cavities on nano scale
- promising candidates for an experimental realization

possible candidates

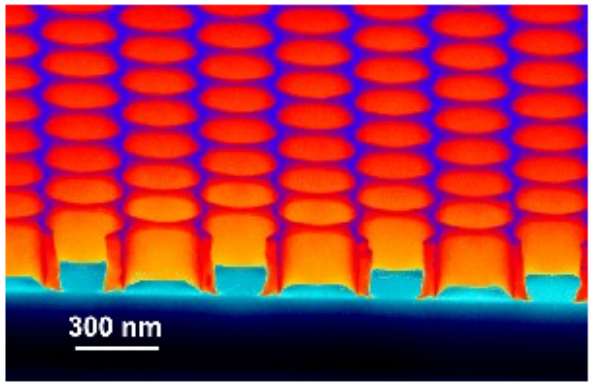
- photonic band gap cavities
- micro-discs and micro-toroids
- fibre based cavities
- on-chip Fabry-Perot cavities
- superconducting stripe-line resonators

Photonic band gap cavities





Defects in photonic crystal structures doped with atoms or q-dots (M. Atature, Cambridge)

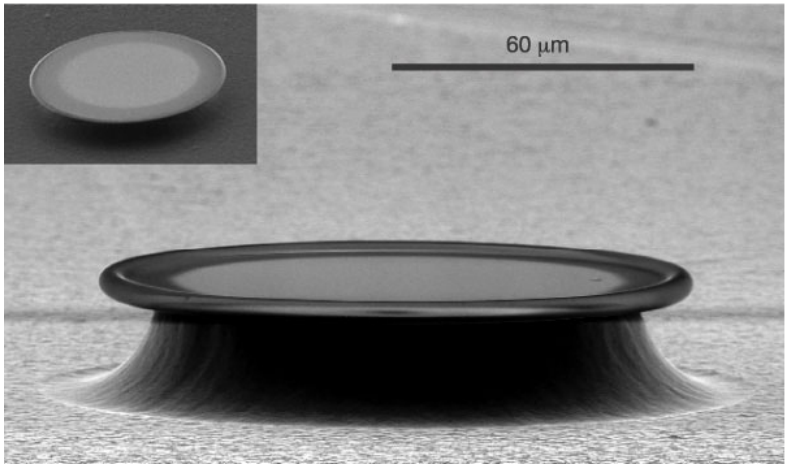


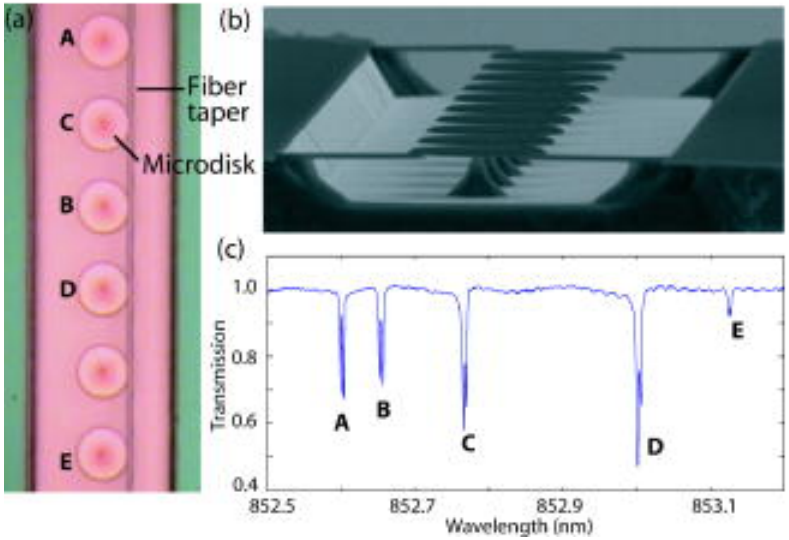
Photonic crystal composed of a periodic array of holes etched in silicon slab. False-colored SEM image.

fig. left from Hartmann, Brandao, Plenio; Laser and Photon. Rev. 2, No. 6, 527-556 (2008); fig. right from IBM

- structures with periodic dielectric properties and band gaps in frequency space
- pro
  - large arrays; small volume $\Longrightarrow$ efficient tunable coupling
- contra
  - hard to produce large arrays with highly periodic defects

Micro-discs and micro-toroids

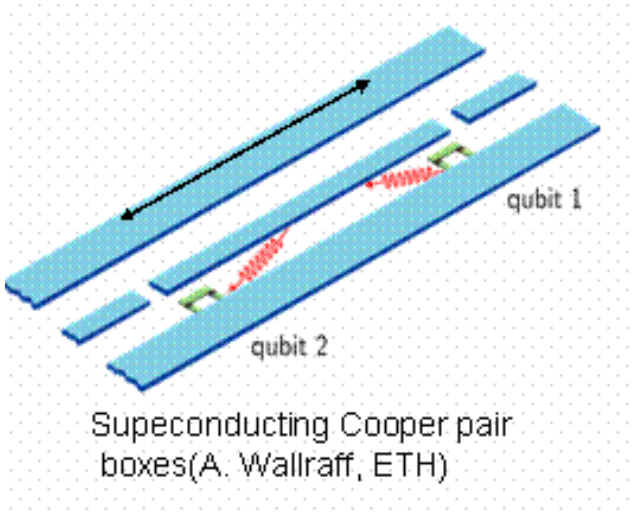




- pro
  - routinely produced in large arrays
  - small volume $\Longrightarrow$ efficient tunable coupling
- contra
  - need to trap atom close to surface for a long time
  - cavities need to be tuned in resonance with each other

- Thanks for your attention!

Superconducting stripe-line resonators



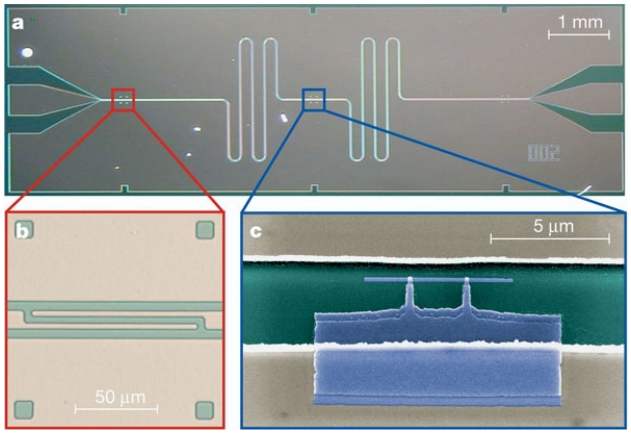
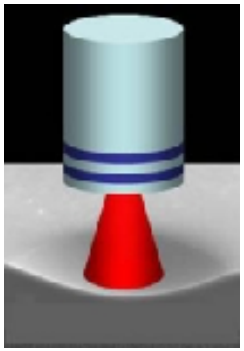
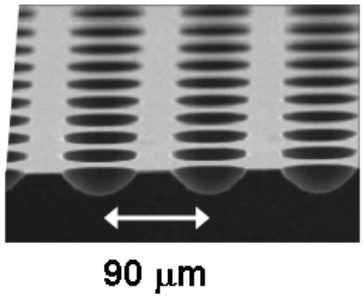


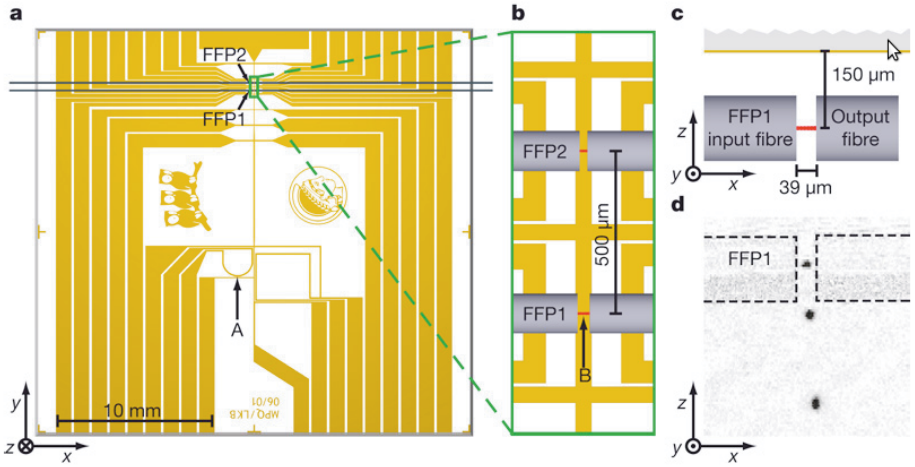
fig. right from Hartmann, Brandao, Plenio; Laser and Photon. Rev. 2, No. 6, 527-556 (2008)

- pro
  - strong coupling; operates in microwave regime
- contra
  - just very small arrays possible up to now
  - quasi one dimensional

On-chip Fabry-Perot and fibre based cavities







left/middle: on-chip Fabry-Perot cavities; right: FFP chip [Hartmann et al.; Laser and Photon, Rev.2, No.6, (2008)]

- pro
  - very small volume $\Longrightarrow$ strong coupling
  - tunable hopping strength over distance of fibres
- contra
  - hopping modification due to photons localised in fibre
  - trapping atoms for sufficient long periods of time

action of $\hat{a}$, $\hat{a}^\dagger$ on the polariton state

- problem: action of $\hat{a}$, $\hat{a}^\dagger$ on n-polariton state unknown
- basic idea:

$$\hat{a}_j \begin{pmatrix} |n+\rangle \\ |n-\rangle \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} t_{n++} & t_{n+-} \\ t_{n-+} & t_{n--} \end{pmatrix} \begin{pmatrix} |n-1,+\rangle \\ |n-1,-\rangle \end{pmatrix}$$

- using the definition of $|n,+\rangle$, $|n,-\rangle$ and the action of $\hat{a}$, $\hat{a}^\dagger$ on the Fock states we find

transition amplitudes

$$t_{n\pm-} = \sqrt{n}\, a_{n\pm}\, b_{n-1+} + \sqrt{n-1}\, b_{n\pm}\, b_{n-1-}$$
$$t_{n\pm+} = \sqrt{n}\, a_{n\pm}\, a_{n-1+} + \sqrt{n-1}\, b_{n\pm}\, a_{n-1-}$$

$$a_{n\pm} = \begin{cases} \sin\theta_n\,, & + \\ \cos\theta_n\,, & - \end{cases} \qquad b_{n\pm} = \begin{cases} \cos\theta_n\,, & + \\ -\sin\theta_n\,, & - \end{cases}$$

Polariton mapping

- its more convenient to use operator representation
- define projection operators for polariton states

Polariton projection operator

$$\hat{P}^{\dagger}_{jn\alpha} = |n\alpha\rangle_j \langle 0-|_j \ , \quad \hat{P}_{jn\alpha} = |0-\rangle_j \langle n\alpha|_j$$

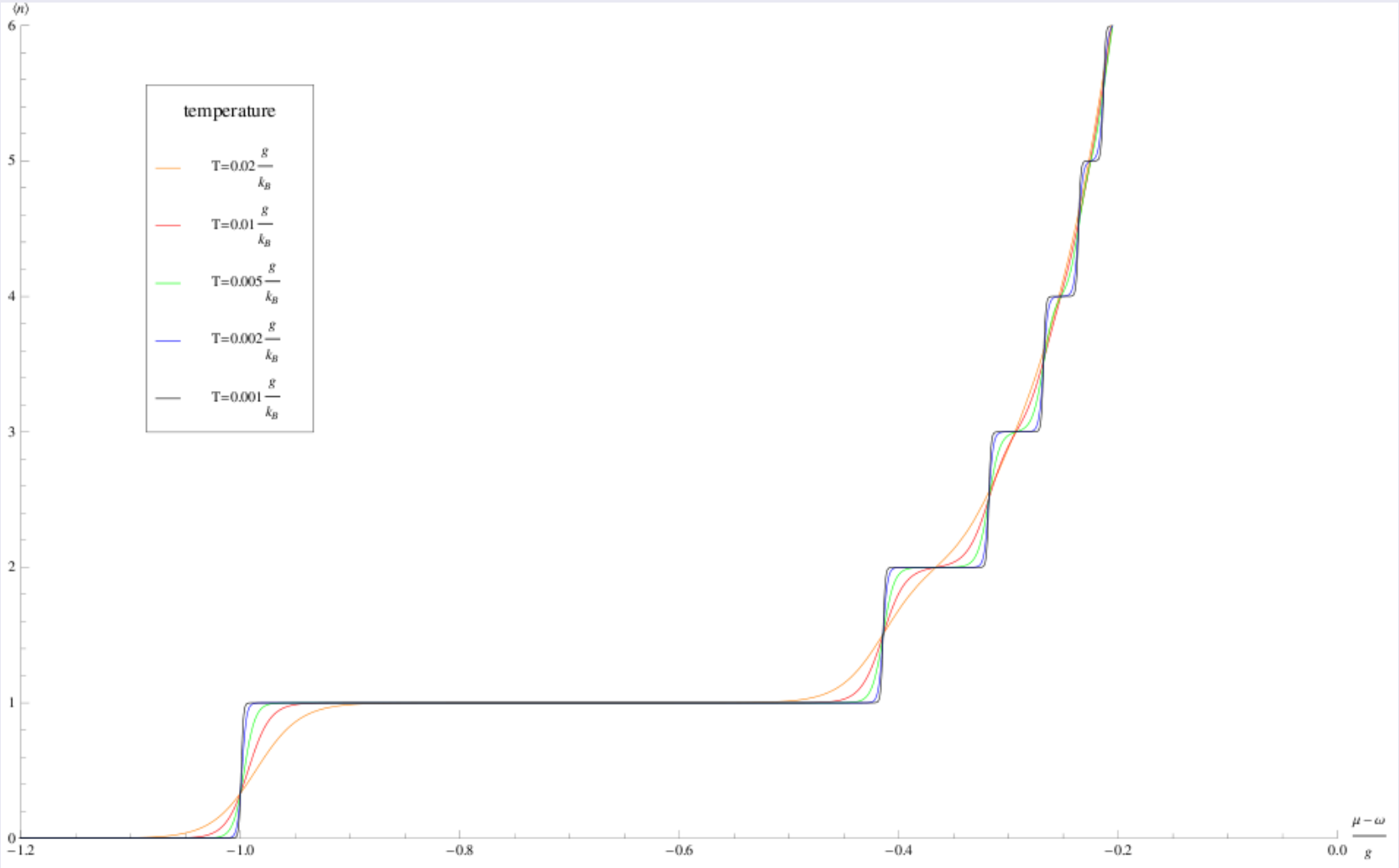
- rewriting $\hat{a}$, $\hat{a}^{\dagger}$ in terms of these projection operators yields

$\hat{a}$, $\hat{a}^{\dagger}$ in polariton picture

$$\hat{a}_j = \sum_{n=1}^{\infty} \sum_{\alpha\alpha'} t_{n\alpha\alpha'} \hat{P}^{\dagger}_{j(n-1)\alpha'} \hat{P}_{jn\alpha}$$
$$\hat{a}_j^{\dagger} = \sum_{n=0}^{\infty} \sum_{\alpha\alpha'} t_{(n+1)\alpha'\alpha} \hat{P}^{\dagger}_{j(n+1)\alpha'} \hat{P}_{jn\alpha}$$

see J.Koch, K.Le Hur, Phys. Rev. A 80, 023811 (2009)

On site polariton number



Jaynes-Cummings eigenvalues

- Hamiltonian separates into 2-dimensional subspaces

$$\hat{\mathcal{H}}^{\mathrm{JC}} = \sum_{n=1}^{N} \hat{h}_n$$

with $\hat{h}_n$ given in the Fock-space representation as

$$\hat{h}_n = \begin{pmatrix} \omega n & g\sqrt{n} \\ g\sqrt{n} & \omega n + \Delta \end{pmatrix}$$

- eigenvalues for upper and lower polariton branch

energy eigenvalues:

$$E_{n\pm} = \omega n + \frac{1}{2}\left(\Delta \pm \sqrt{\Delta^2 + 4g^2 n}\right), \quad (n > 1), \quad E_0 = 0$$

perturbation part becomes

$$\hat{H}_{1\mathrm{D}}(\tau) = -\sum_{ij} \kappa_{ij} \hat{a}^{\dagger}_{i\mathrm{D}}(\tau) \hat{a}_{j\mathrm{D}}(\tau) + \sum_{i} \left[j_i^*(\tau) \hat{a}_{i\mathrm{D}}(\tau) + j_i(\tau) \hat{a}^{\dagger}_{i\mathrm{D}}(\tau) \right]$$

- inserting the above expression into the partition function and expanding yields

functional partition function

$$\mathcal{Z} = \mathcal{Z}_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^{\beta} d\tau_1 \ldots \int_0^{\beta} d\tau_n \left\langle \hat{T} \left[\hat{H}_{1\mathrm{D}}(\tau_1) \ldots \hat{H}_{1\mathrm{D}}(\tau_n) \right] \right\rangle_0$$

- where the averages correspond to n-particle Green's functions