Bosons in Optical Lattices

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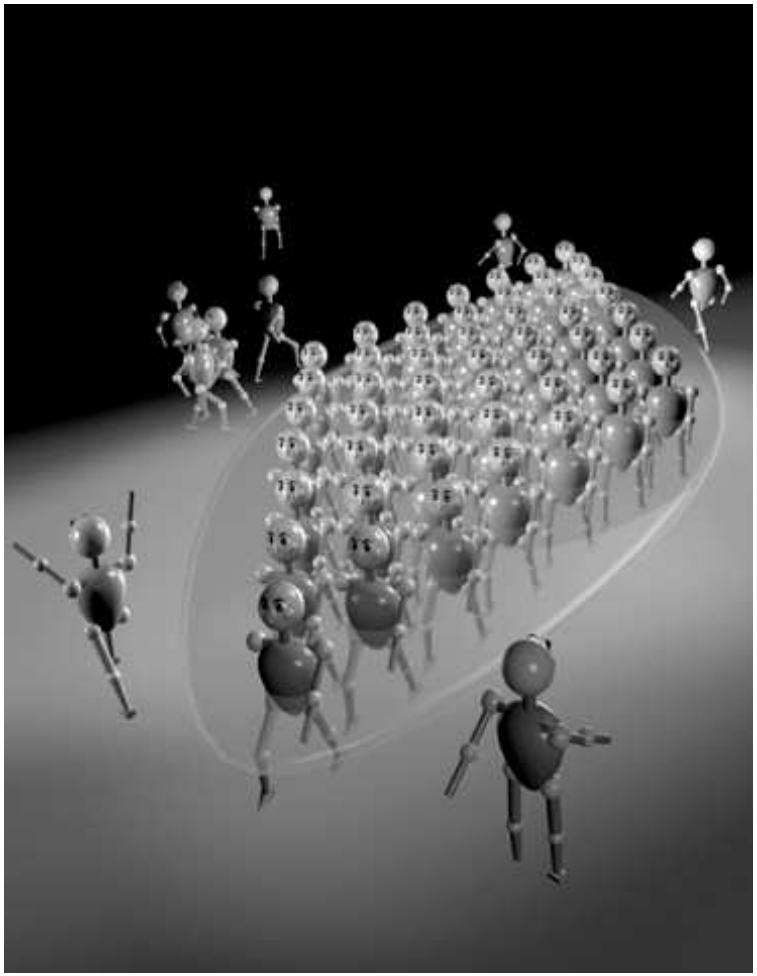
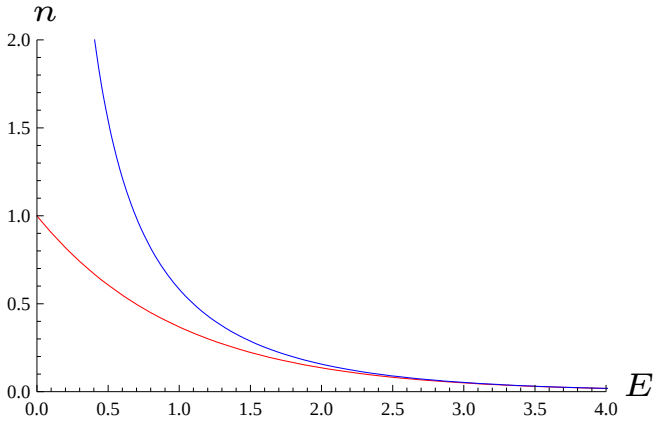
Content



Science Cover, December 1995

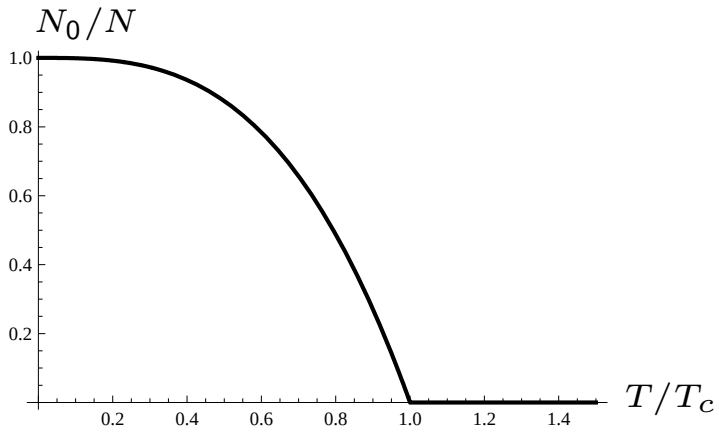
Theoretical Prediction

- Predicted by Bose and Einstein for ideal gas of Bosons (1924)
- Macroscopic occupation of ground state
- Purely statistical effect, no interaction involved
- Connected to suprafluid Helium



Red: Classical. Blue: Bose-Einstein

Ground-state occupation:



$$\lambda_c = \sqrt{\frac{2\pi\hbar^2}{Mk_BT_c}} \approx n^{-1/3}$$

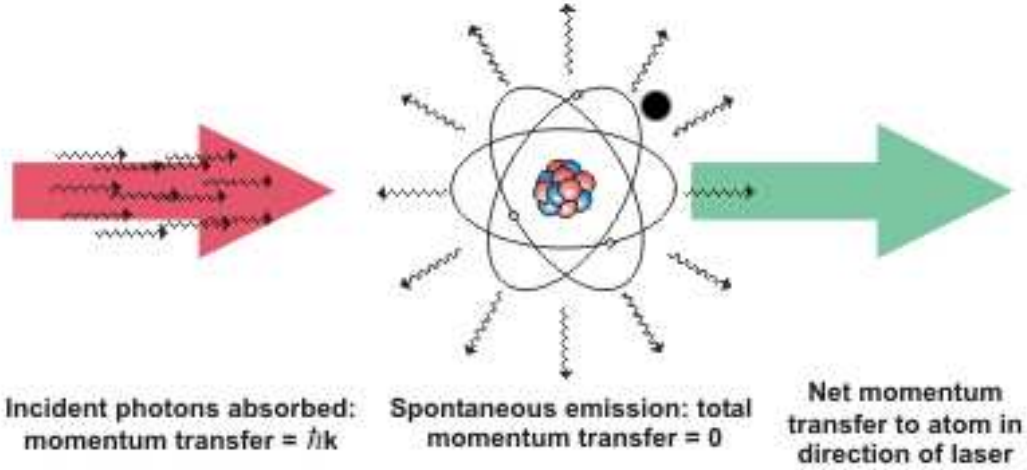
Critical temperature:

$$T_c \approx 0.08\frac{h^2}{Mk_B}n^{2/3}$$

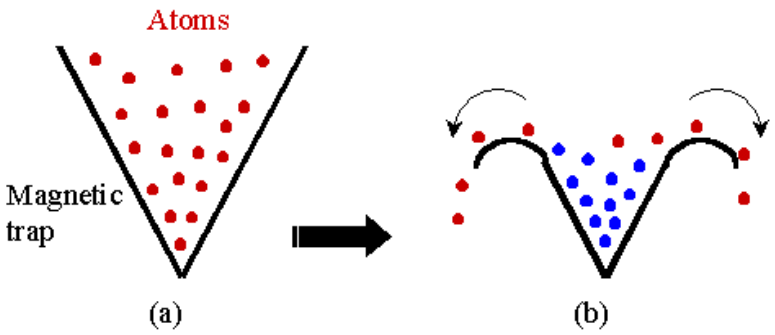
Cooling Techniques

- BEC only possible in very dilute gases to avoid "freezing"
- Nano-Kelvin temperatures necessary to reach BEC
- Various cooling methods applied successively

Laser cooling:



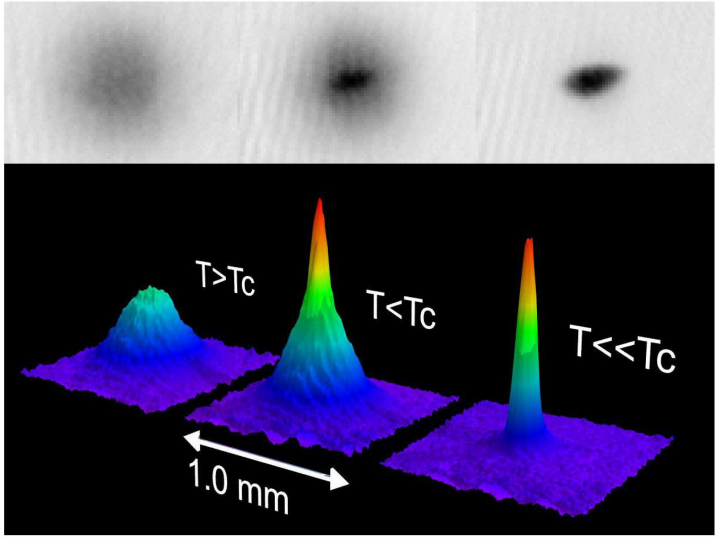
Evaporative cooling:



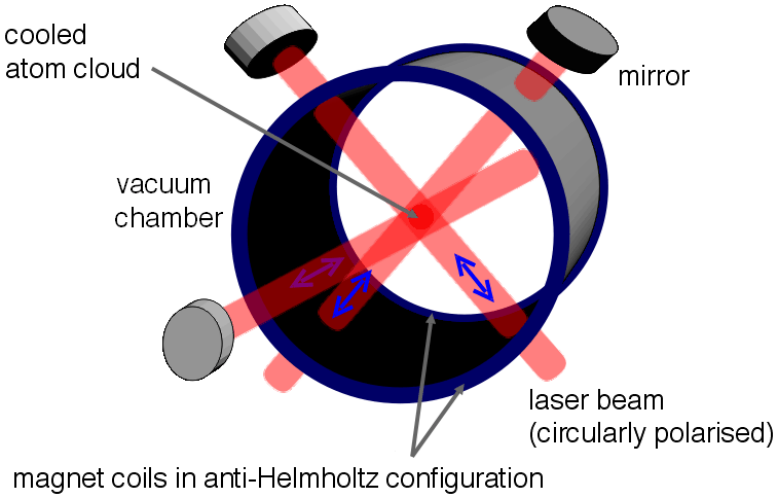
Experimental Observation

- First observed 1995 at JILA and MIT in Rubidum and Sodium
- $6 \cdot 10^5$ atoms, $T_c \approx 250\,nK$

Time-of-flight absorption image

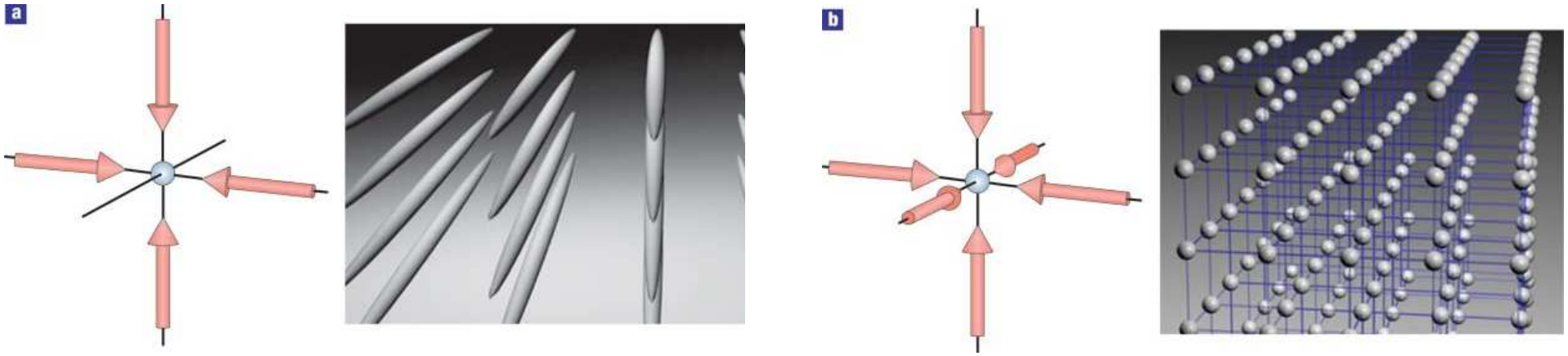


Magneto-optical trap (MOT)



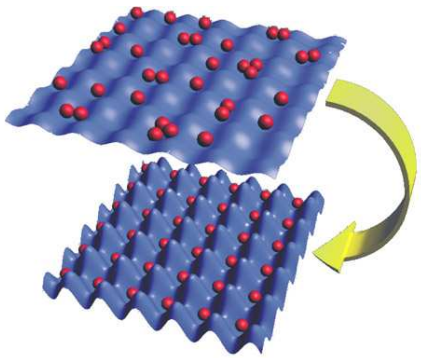
Principles of Optical Lattices

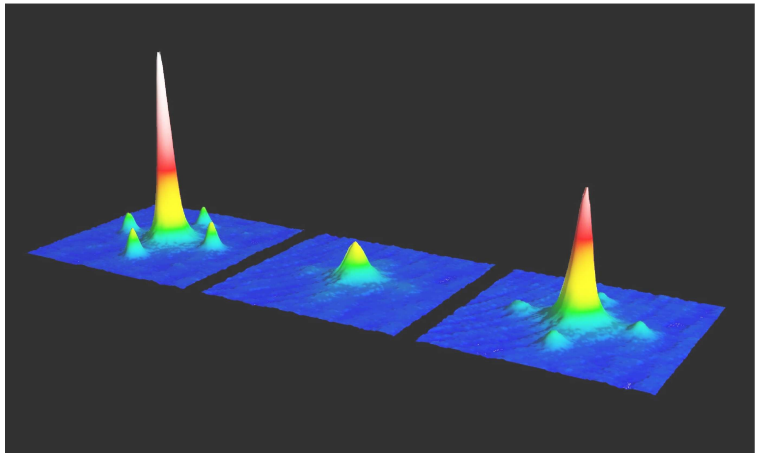
- Optical lattices produced by counter-propagating lasers
- $V = V_0 \sum_{i=1}^{D} \sin^2(2\pi x_i/\lambda)$
- Relative strength of hopping and interaction controllable
- (Quasi) one-, two-, and three-dimensional configurations possible
- Model system for condensed matter physics

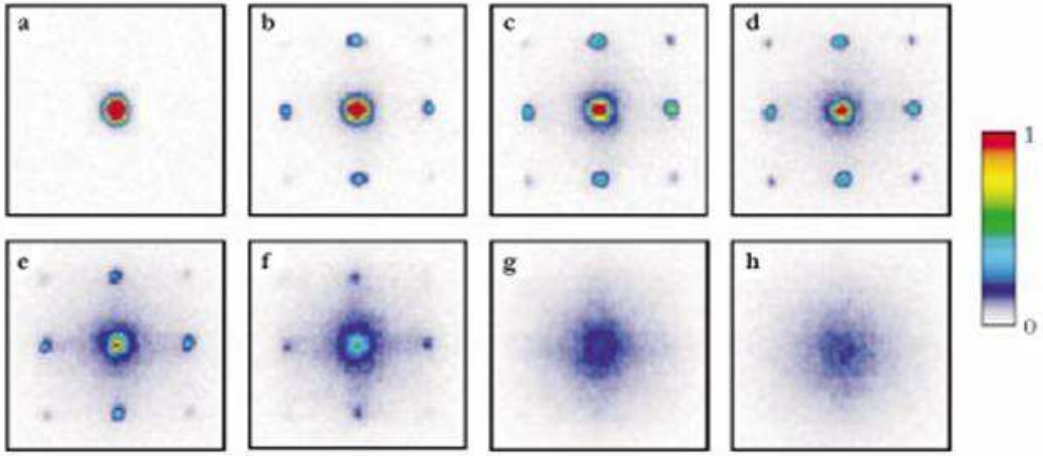


Realization of Superfluid-Mott insulator transition

- Increasing the laser intensity drives transition from delocalized to localized state
- Experimentally detectable in time-of-flight pictures

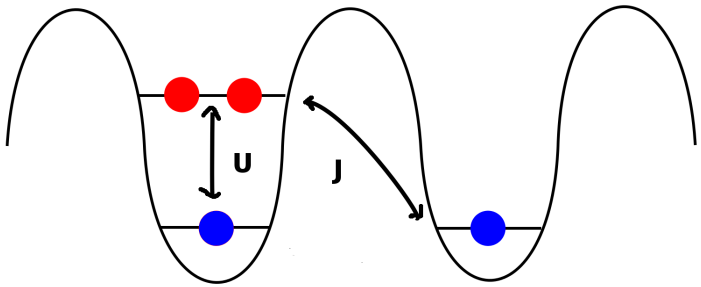






Bose-Hubbard Model

- Bose-Hubbard Hamiltonian:



$$\hat{H}_{\mathrm{BHM}} = \hat{H}_0 + \hat{H}_1$$

$$\hat{H}_0 = \sum_i \left[\frac{U}{2} \hat{n}_i (\hat{n}_i - 1) - \mu \hat{n}_i \right]$$

$$\hat{H}_1 = -J \sum_{<i,j>} \hat{a}_i^\dagger \hat{a}_j = -\sum_{i,j} J_{i,j} \hat{a}_i^\dagger \hat{a}_j$$

$$\hat{n}_i = \hat{a}_i^\dagger \hat{a}_i$$

$$J_{ij} = \begin{cases} J & \text{if } i, j \text{ nearest neighbors} \\ 0 & \text{otherwise} \end{cases}$$

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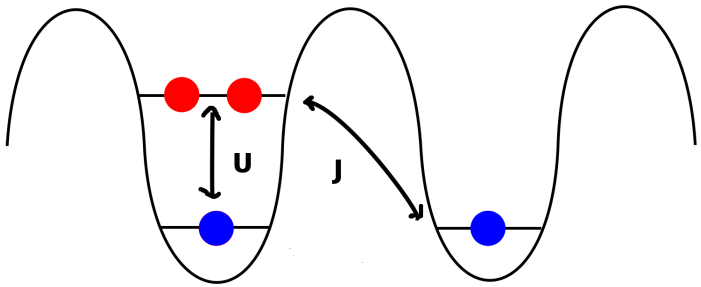

- $\hat{H}_0$ site-diagonal

$$\hat{H}_0 |n\rangle = N_S E_n |n\rangle$$

$$E_n = \frac{U}{2} n(n-1) - \mu n$$

- Perturbative expansion in $\hat{H}_1$

Imaginary-Time Green's Function

- Definition:

$$G_1(\tau', j'|\tau, j) = \frac{1}{\mathcal{Z}} \mathrm{Tr}\left\{ e^{-\beta \hat{H}} \hat{T}\left[\hat{a}_{j,\mathrm{H}}(\tau) \hat{a}^{\dagger}_{j',\mathrm{H}}(\tau') \right] \right\}$$

$$\text{with } \mathcal{Z} = \mathrm{Tr}\{e^{-\beta \hat{H}}\}$$

- Heisenberg operators in imaginary time ($\hbar = 1$):

$$\hat{X}_{\mathrm{H}}(\tau) = e^{-\hat{H}\tau} \hat{X} e^{\hat{H}\tau}$$

Dirac Interaction Picture

- Time evolution of the operators determined only by $\hat{H}_0$:

$$\hat{O}_{\mathrm{D}}(\tau) = e^{\hat{H}_0 \tau} \hat{O} e^{-\hat{H}_0 \tau}$$

- Dirac time-evolution operator calculated by Dyson series:

$$\begin{aligned}\hat{U}_{\mathrm{D}}(\tau, \tau_0) =& \sum_{n=0}^{\infty} (-1)^n \int_{\tau_0}^{\tau} d\tau_1 \ldots \int_{\tau_0}^{\tau_{n-1}} d\tau_n \hat{H}_{1\mathrm{D}}(\tau_1) \ldots \hat{H}_{1\mathrm{D}}(\tau_n) \\ =& \hat{T} \exp\left(- \int_{\tau_0}^{\tau} d\tau_1 \hat{H}_{1\mathrm{D}}(\tau_1) \right)\end{aligned}$$

Partition Function

- Full partition function:

$$\mathcal{Z} = \mathrm{Tr}\left\{e^{-\beta \hat{H}_0} \hat{U}_{\mathrm{D}}(\beta, 0)\right\}$$

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- nth order contribution:

$$\mathcal{Z}^{(n)} = \frac{1}{n!}\mathcal{Z}^{(0)}\sum_{i_1,j_1,\ldots,i_n,j_n} J_{i_1 j_1}\ldots J_{i_n j_n}\int_0^\beta d\tau_1\int_0^\beta d\tau_2\ldots\int_0^\beta d\tau_n$$
$$\times\, G_n^{(0)}(\tau_1, j_1;\ldots;\tau_n, j_n|\tau_1, i_1;\ldots;\tau_n, i_n)$$

- Unperturbed n-particle Green's function:

$$G_n^{(0)}(\tau_1', i_1';\ldots;\tau_n', i_n'|\tau_1, i_1;\ldots;\tau_n, i_n) = \left\langle \hat{T}\hat{a}^\dagger_{i_1'}(\tau_1')\hat{a}_{i_1}(\tau_1)\ldots\hat{a}^\dagger_{i_n'}(\tau_n')\hat{a}_{i_n}(\tau_n)\right\rangle^{(0)}$$

Cumulant Decomposition

- Decompose $G_n^{(0)}(\tau_1', i_1'; \ldots; \tau_n', i_n' | \tau_1, i_1; \ldots; \tau_n, i_n)$ into "simple" parts
- $\hat{H}_0$ not harmonic $\Rightarrow$ Wick's theorem not applicable

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- But: Decomposition into cumulants
- $\hat{H}_0$ site-diagonal $\Rightarrow$ cumulants local. Example:

$$G_2^{(0)}(\tau_1', i_1'; \tau_2', i_2' | \tau_1, i_1; \tau_2, i_2) = \delta_{i_1,i_2}\delta_{i_1',i_2'}\delta_{i_1,i_1'} C_2^{(0)}(\tau_1', \tau_2' | \tau_1, \tau_2)$$
$$+ \delta_{i_1,i_1'}\delta_{i_2,i_2'} C_1^{(0)}(\tau_1' | \tau_1) C_1^{(0)}(\tau_2' | \tau_2) + \delta_{i_1,i_2'}\delta_{i_2,i_1'} C_1^{(0)}(\tau_2' | \tau_1) C_1^{(0)}(\tau_1' | \tau_2)$$

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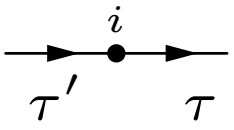
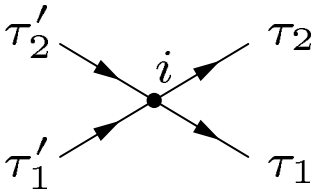
- Denote contributions diagrammatically. Points for cumulants, lines for hopping matrix elements
- Perturbation theory in number of *lines*

Diagrammatic Rules for $\mathcal{Z}^{(n)}$

1. Draw all possible combinations of vertices with total n entering and leaving lines
2. Connect them in all possible ways and assign time variables and hopping matrix elements onto the lines
3. Sum all site indices and integrate all time variables from 0 to β



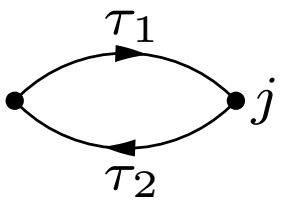
$= C_1^{(0)}(\tau'|\tau)$



$= C_2^{(0)}(\tau_1', \tau_2'|\tau_1, \tau_2)$

$= J_{ij}$

example : $\mathcal{Z}^{(2)} = \frac{1}{2}$



Diagrammatic Rules for Green's Function

$$G_1(\tau', i'|\tau, i) = \frac{1}{\mathcal{Z}} \mathsf{Tr}\left\{ e^{-\beta \hat{H}_0} \hat{T} \hat{a}^\dagger_{i'}(\tau') \hat{a}_i(\tau) \hat{U}_\mathsf{D}(\beta, 0) \right\}$$

$$G_1^{(n)}(\tau', i'|\tau, i) = \frac{\mathcal{Z}^{(0)}}{\mathcal{Z}} \frac{1}{n!} \sum_{i_1, j_1, \ldots, i_n, j_n} J_{i_1 j_1} \ldots J_{i_n j_n} \int_0^\beta d\tau_1 \ldots \int_0^\beta d\tau_n$$
$$\times G^{(0)}_{n+1}(\tau_1, j_1; \ldots; \tau_n, j_n; \tau', i'|\tau_1, i_1; \ldots; \tau_n, i_n, \tau, i)$$

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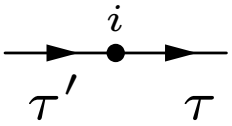
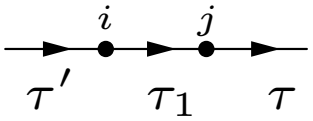
- Zeroth and first order:

$G_1^{(0)}(\tau', i|\tau, j) =$ 

$= \delta_{i,j} C_1^{(0)}(\tau'|\tau)$

$G_1^{(1)}(\tau', i|\tau, j) =$ 

$= J \delta_{d(i,j),1} \int_0^\beta d\tau_1 \, C_1^{(0)}(\tau'|\tau_1) C_1^{(0)}(\tau_1|\tau)$

Calculations in Matsubara Space

- Translational invariance in time suggests Matsubara transform

$$C_1^{(0)}(\omega_m) = \frac{1}{\mathcal{Z}^{(0)}} \sum_n \left[\frac{(n+1)}{E_{n+1} - E_n - i\omega_m} - \frac{n}{E_n - E_{n-1} - i\omega_m} \right] e^{-\beta E_n}, \ \omega_m = \frac{2\pi}{\beta} m$$

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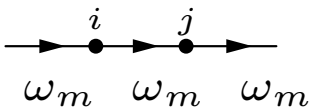
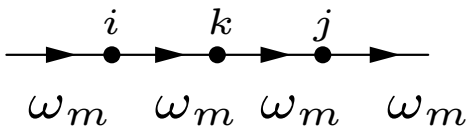
- In rule 3 integration over τ replaced by summation over ω_m under consideration of frequency conservation on vertices:

$$G_1^{(1)}(\omega_m; i, j) =$$



$$= J\delta_{d(i,j),1} C_1^{(0)}(\omega_m)^2$$

$$G_2^{(1)}(\omega_m; i, j) =$$



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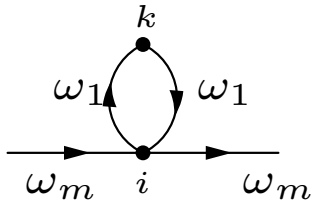
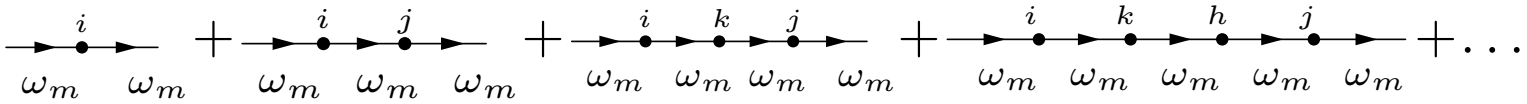


$$= J^2(\delta_{d(i,j),2} + 2\delta_{d(i,j),\sqrt{2}} + 2D\delta_{i,j}) C_1^{(0)}(\omega_m)^3$$
$$+ J^2 2D\delta_{i,j} \sum_{\omega_1} C_1^{(0)}(\omega_m) C_2^{(0)}(\omega_m, \omega_1 | \omega_m, \omega_1)$$

First-Order Resummation

- Improvement of perturbation theory by resummation

$\tilde{G}_1(i,\omega_m|j) =$ 

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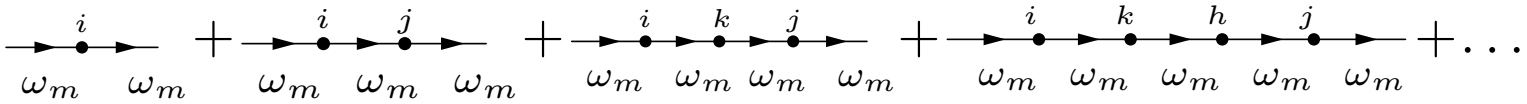
$\tilde{G}_1(i,\omega_m|j) =$ 

- Summed most easily in Fourier space:

$$\tilde{G}_1^{(1)}(\omega_m,\mathbf{k}) = \frac{C_1^{(0)}(\omega_m)}{1 - J(\mathbf{k})\, C_1^{(0)}(\omega_m)}\,, \qquad J(\mathbf{k}) = 2J\sum_{l=1}^{D}\cos(k_l a)$$

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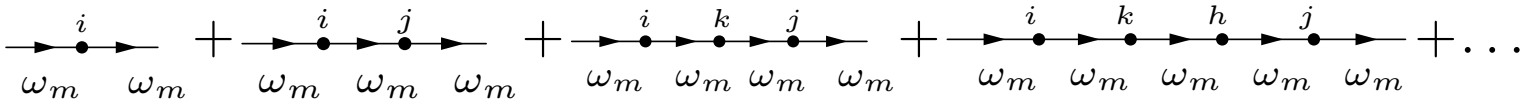
$\tilde{G}_1(i,\omega_m|j) =$ 

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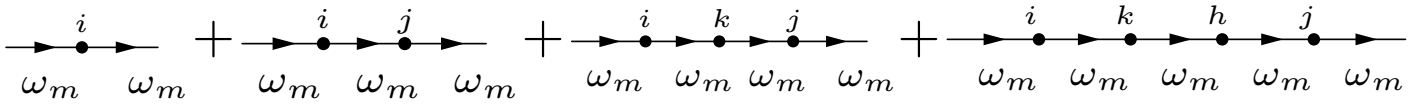
$\tilde{G}_1(i,\omega_m|j) =$ 

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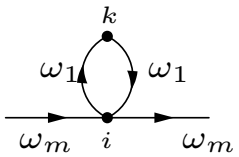
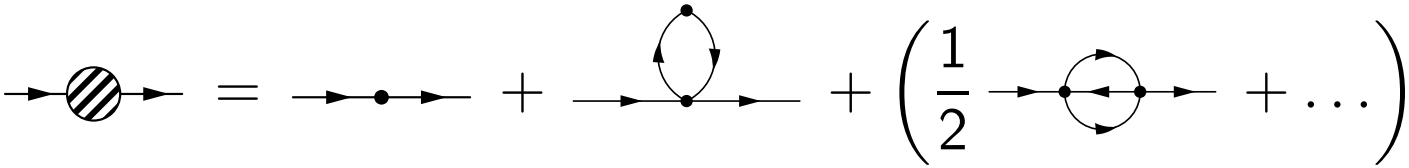
$$\tilde{G}_1^{(1)}(\omega_m,\mathbf{k}) = \frac{C_1^{(0)}(\omega_m)}{1 - J(\mathbf{k})\, C_1^{(0)}(\omega_m)}\,, \qquad J(\mathbf{k}) = 2J\sum_{l=1}^{D}\cos(k_l a)$$

- Neglected contributions like 

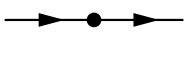
vanish at least as $1/D$ for $D \to \infty$

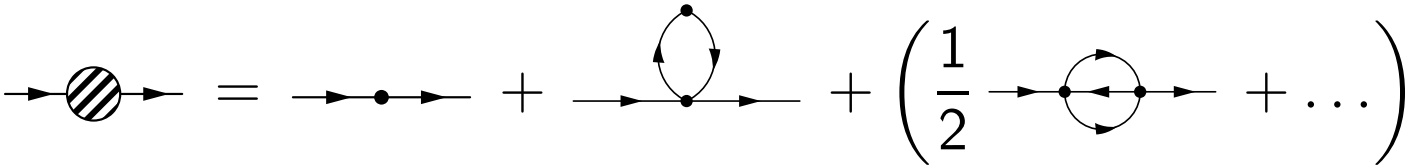
General Resummation Technique

- Replace  by sum over all one-particle irreducable diagrams:



General Resummation Technique

- Replace  by sum over all one-particle irreducable diagrams:



- Full Green's function obtained by

$$G_1(\omega_m, \mathbf{k}) = \sum_{l=0}^{\infty} \Big(\Big)^{l+1} J(\mathbf{k})^l$$

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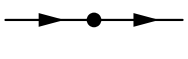
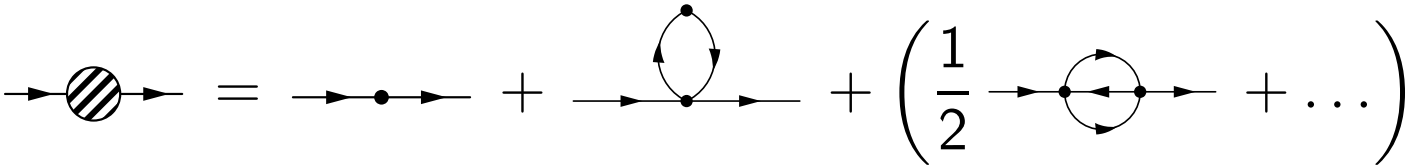
- Replace  by sum over all one-particle irreducable diagrams:



- Full Green's function obtained by

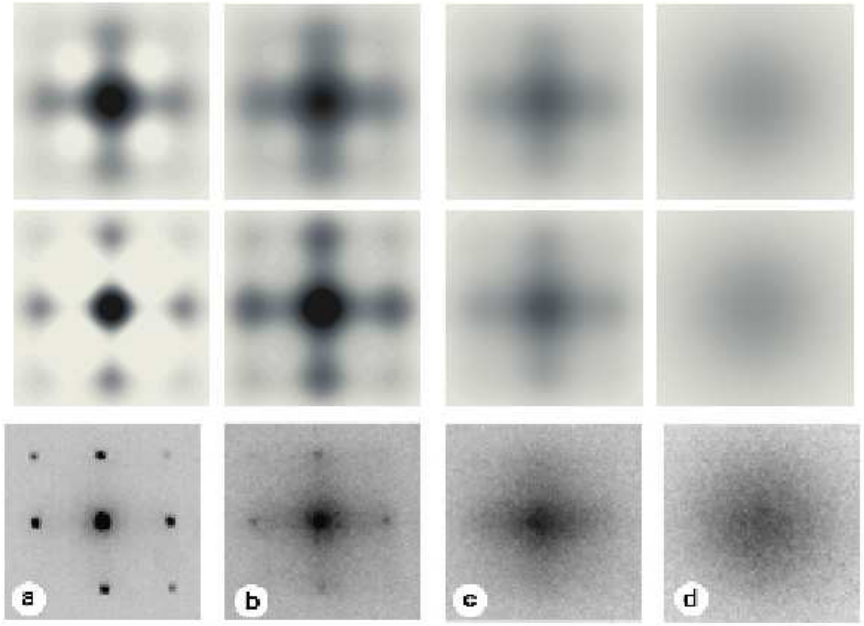
$$G_1(\omega_m, \mathbf{k}) = \sum_{l=0}^{\infty} \Big(\Big)^{l+1} J(\mathbf{k})^l$$

- One-loop approximation by considering only the first two terms in 

Time-of-Flight Pictures for $T = 0$

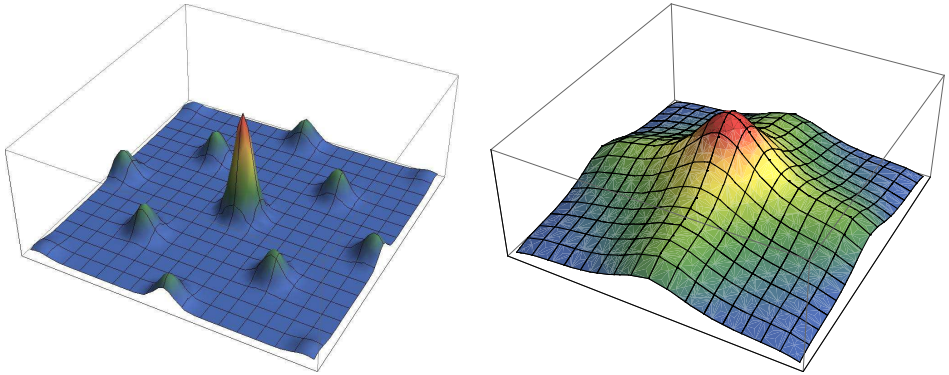
Momentum space density:

$$n_{\mathbf{k}} = \langle \hat{\psi}^\dagger(\mathbf{k}) \hat{\psi}(\mathbf{k}) \rangle = |w(\mathbf{k})|^2 S(\mathbf{k}), \quad S(\mathbf{k}) = \sum_{i,j} e^{i\mathbf{k}(\mathbf{r}_i - \mathbf{r}_j)} \lim_{\tau' \searrow 0} G_1(\tau', i|0, j)$$



Top to bottom: 1st order, 2nd order, experiment
Left to right: $V_0 = 8\,E_R$, $V_0 = 14\,E_R$
$V_0 = 18\,E_R$, $V_0 = 30\,E_R$

Resummed Greens function yields sharp peaks in superfluid phase:



$V_0 = 8\,E_R$ $\qquad$ $V_0 = 18\,E_R$

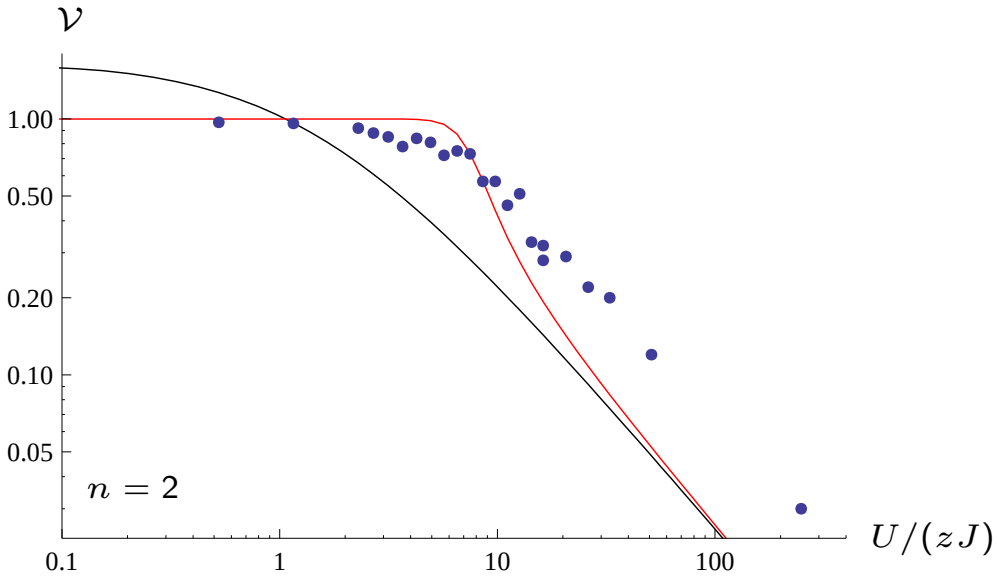
Recoil energy: $E_R = \frac{h^2}{2m\lambda^2}$

Visibility

- Visibility allows quantitative discussion of TOF-Pictures

$$\mathcal{V} = \frac{n_{\mathsf{max}} - n_{\mathsf{min}}}{n_{\mathsf{max}} + n_{\mathsf{min}}}$$

- Expected to converge to unity for $U = 0$ at $T = 0$
- Reduced by thermal fluctuations



Black: First-order perturbation theory.

Red: First-order resummed.

Dots: Experimental Data, Gerbier, et. al PRA **72**, 053606 (2005)

First-Order Phase Diagram

- Phase boundary given by $\tilde{G}_1^{(1)}(0,\mathbf{0}) \to \infty$

$$2DJ_c = \frac{\sum_n e^{-\beta E_n}}{\sum_n e^{-\beta E_n}\left(\frac{n+1}{E_{n+1}-E_n} - \frac{n}{E_n - E_{n-1}}\right)} \xrightarrow{T\to 0} = \frac{1}{\frac{n+1}{E_{n+1}-E_n} - \frac{n}{E_n - E_{n-1}}}$$

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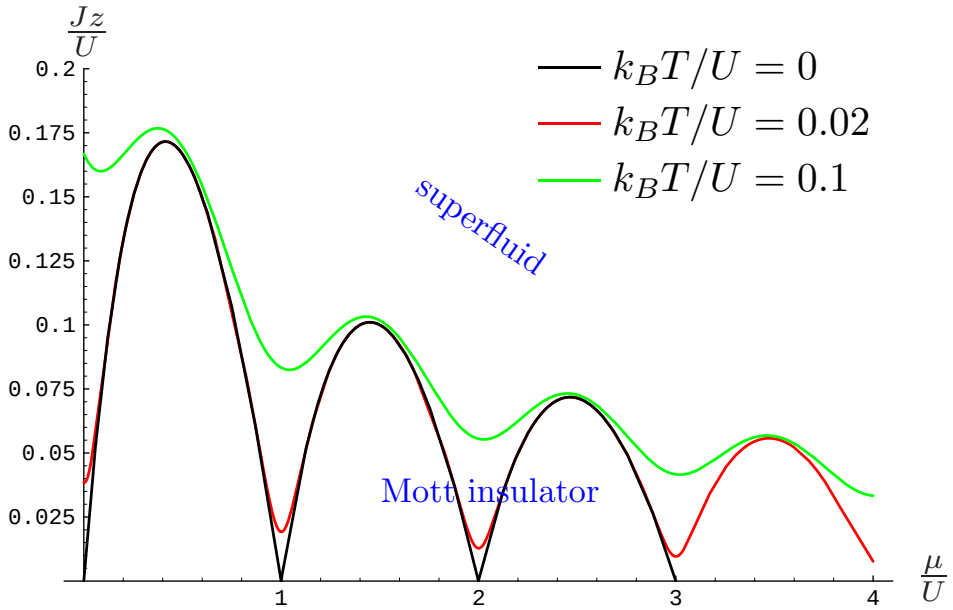
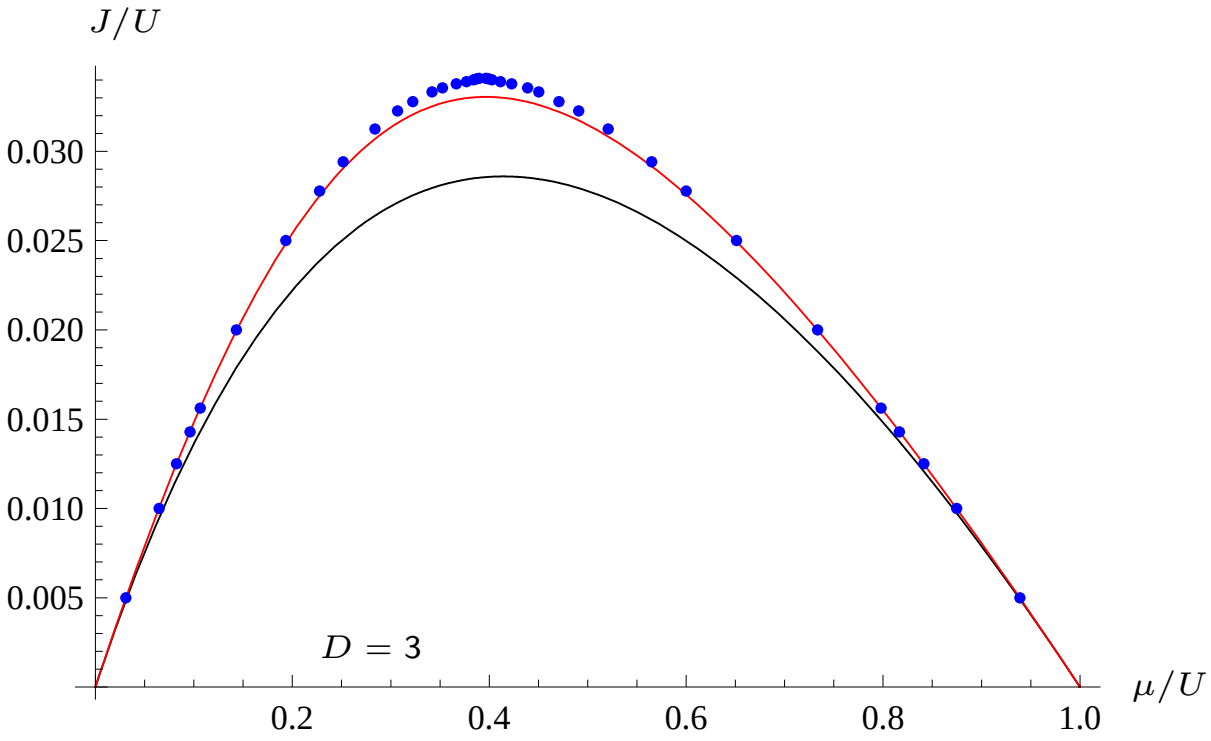
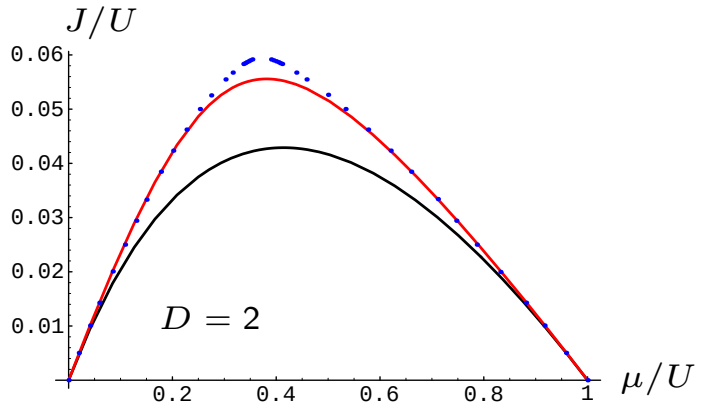
- Same result as obtained by mean-field theory ($z = 2D$)



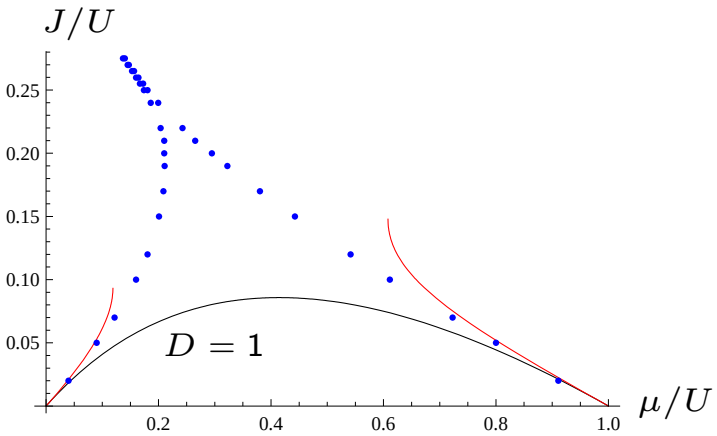
Comparison with Simulations for $T = 0$



Capogrosso-Sansone et. al. PRB **75**, 134302 (2007)



Capogrosso-Sansone et. al. arXiv:0710.2703v3 (2007)

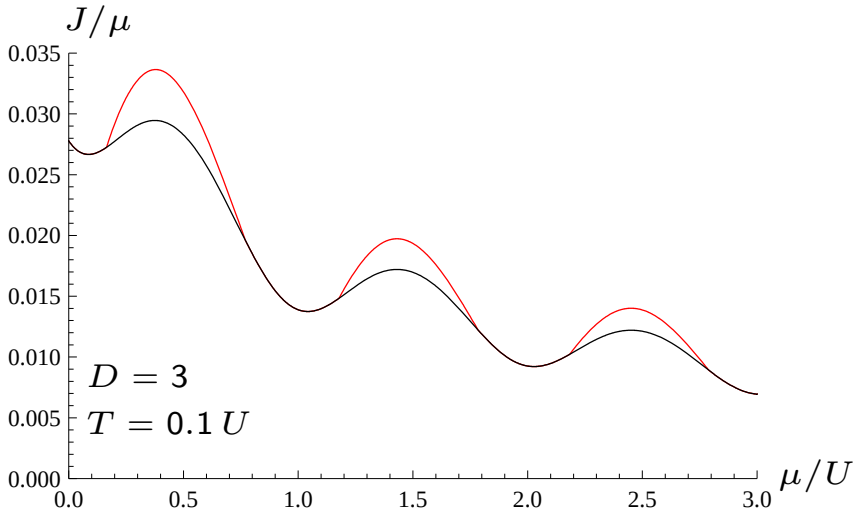


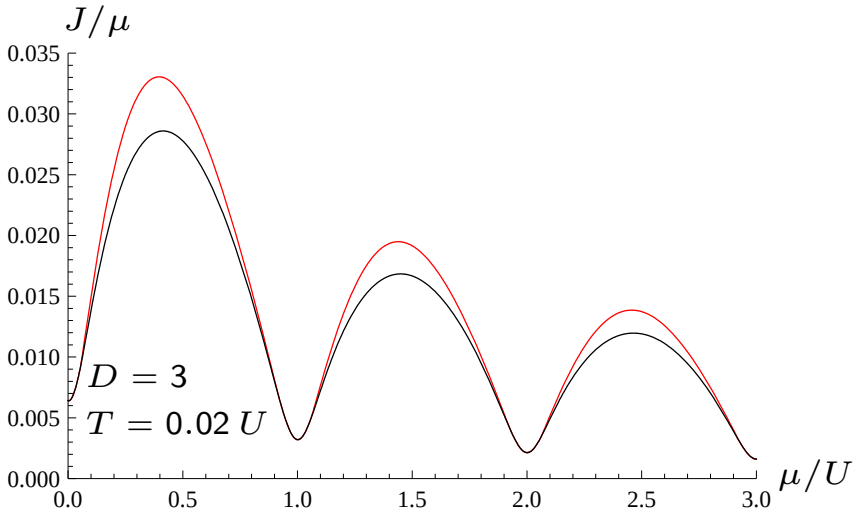
Kühner et. al. PRB **58**, R14741 (1998)

$n = 1$. Black: First order (Mean field)

Red: One-loop corrected

One-Loop Corrected Finite Temperature Phase Diagram





- Temperature effects small at tip of lobe
- One-loop correction largest at zero temperature

Real-Time Green's Function

- Dynamic properties determined by Green's function in real-time:

$$G_1(t', j'|, t, j) = \frac{-i}{\mathcal{Z}} \mathrm{Tr} \left\{ e^{-\beta \hat{H}} \hat{T} \left[\hat{a}_{j,\mathsf{H}}(t), \hat{a}^{\dagger}_{j',\mathsf{H}}(t') \right] \right\}$$

- Can be obtained by analytic continuation of imaginary-time result by replacing

$$\omega_m \longrightarrow -i\omega$$

- Zeroth order:

$$G_1^{(0)}(\omega; i, j) = \frac{-i\delta_{i,j}}{\mathcal{Z}^{(0)}} \sum_n \left[\frac{(n+1)}{E_{n+1} - E_n - \omega} - \frac{n}{E_n - E_{n-1} - \omega} \right] e^{-\beta E_n}$$

Excitation Spectrum

- Excitation spectrum given by poles of real-time Green's function

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- For $T = 0$:

$$\tilde{G}_1^{(1)}(\omega, \mathbf{k}) \stackrel{!}{=} 0$$

$$\Longrightarrow \omega_{1,2} = \frac{U}{2}(2n-1) - \mu - J(\mathbf{k}) \pm \frac{1}{2}\sqrt{U - 2DJ(\mathbf{k})(4n+2) + [2DJ(\mathbf{k})]^2}$$

- Different signs correspond to particle and hole excitations

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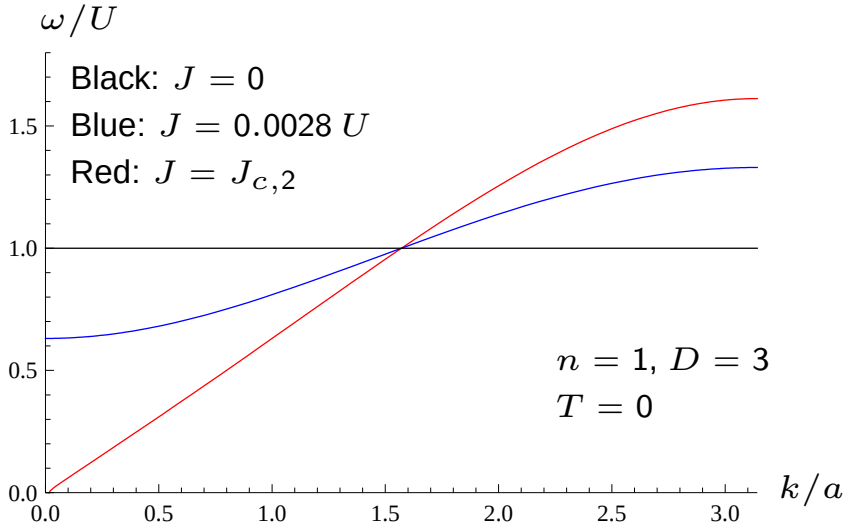
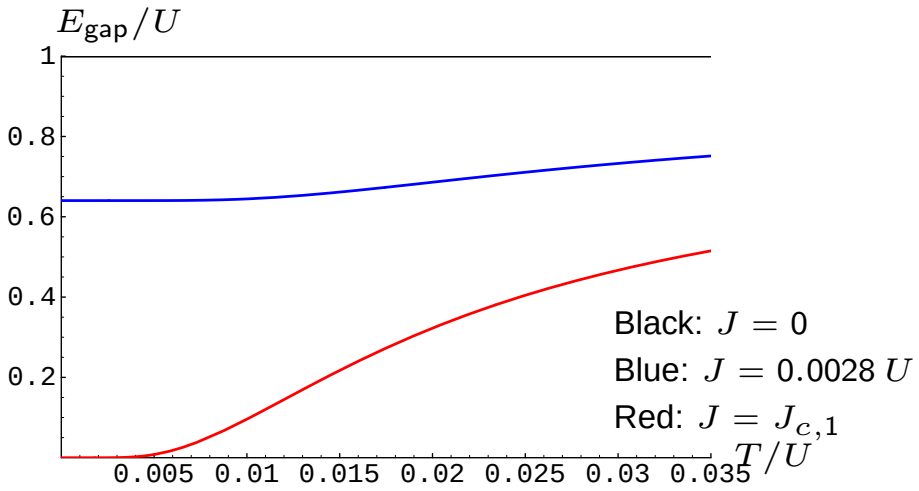
- Different signs correspond to particle and hole excitations
- Dispersion relation of pairs:

$$\omega_{\mathrm{ph}}(\mathbf{k}) = \omega_1(\mathbf{k}) - \omega_2(\mathbf{k}) = \sqrt{U - 2DJ(\mathbf{k})(4n+2) + [2DJ(\mathbf{k})]^2}$$

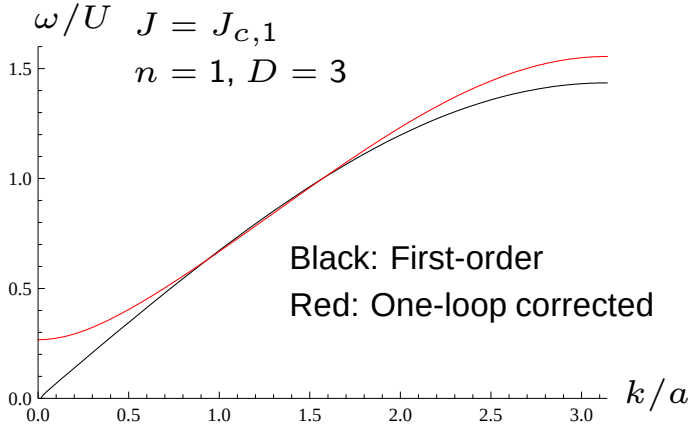
Dispersion Relation



One-loop corrected pairs dispersion relation



Dependence of gap on temperature (first-order)



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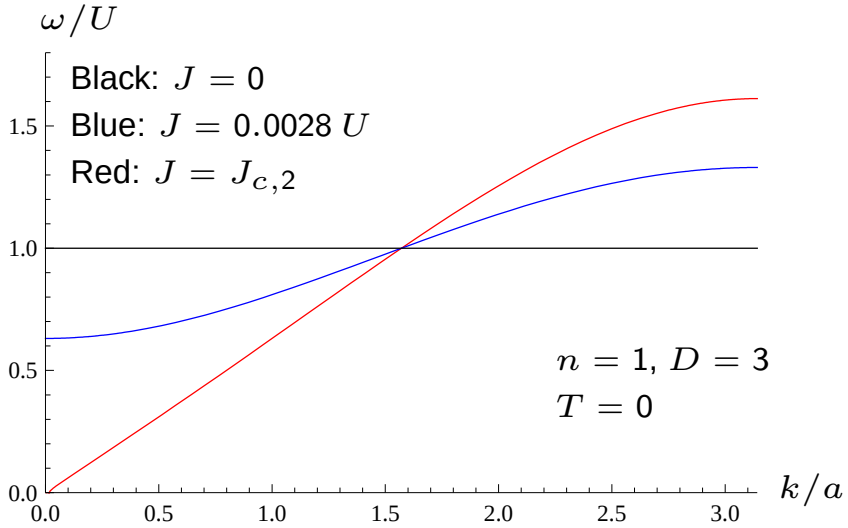

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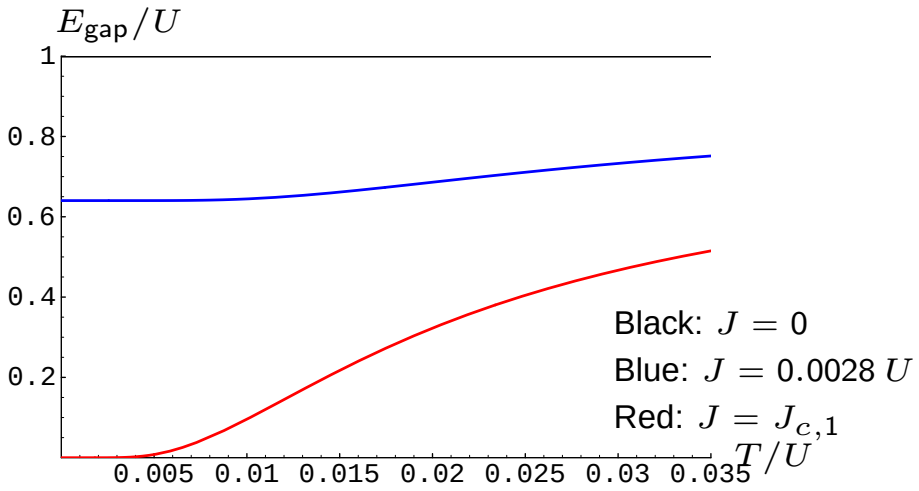

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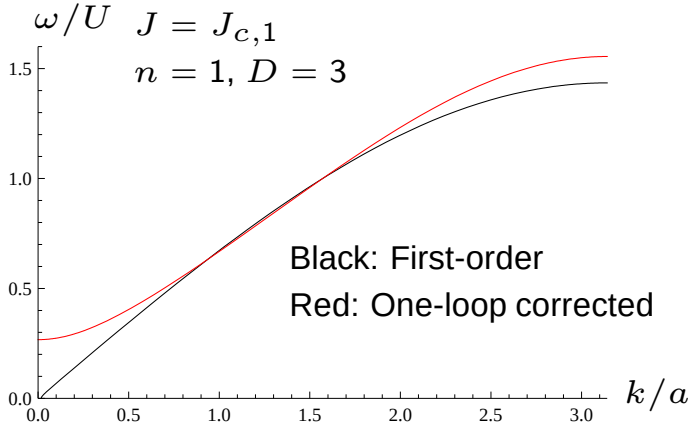


- Gap vanishes at critical hopping J_c
- Gap experimentally measurable, could serve as thermometer

Effective Masses

- Defined by:

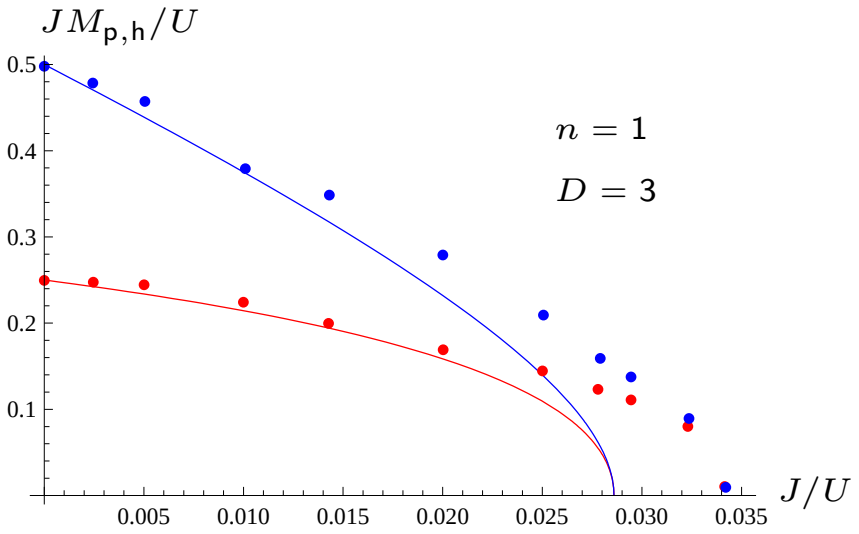
$$\omega_{\mathsf{p,h}}(\mathbf{k}) = E_{\mathsf{gap}} + \frac{\mathbf{k}^2}{2M_{\mathsf{p,h}}} + \dots$$

$$\tilde{J}M_{\mathsf{p,h}} = \frac{\sqrt{1+36\tilde{J}(\tilde{J}-1)}}{3-6\tilde{J}\pm\sqrt{1+36\tilde{J}(\tilde{J}-1)}}$$

- At critical J:
  Excitations become massless

$$\omega_{\mathsf{p,h}}(\mathbf{k}) \propto |\mathbf{k}|$$

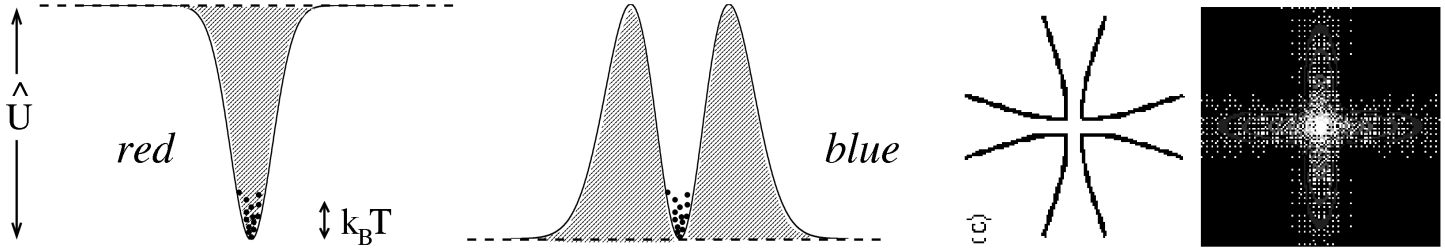
- Particle-hole symmetry at tip of Mott lobe



First-order effective masses of particles (red) and holes (blue). Dots: QMC data. Capogrosso-Sansone et. al. PRB **75**, 134302 (2007)

Optical Trap

- Far detuned lasers induce electric dipole moments
- Electric forces push atoms towards center of trap (Stark effect)
  - Red detuned light ⇒ Atoms are pushed to maximal intensity
  - Blue detuned light ⇒ Atoms are pushed to minimal intensity
- Very shallow, depth smaller than 1mK
- No forced evaporative cooling possible



Spin-1 Bose-Hubbard Hamiltonian

Decomposition of field operators into Wannier functions yields Bose-Hubbard model:

$$\hat{H}_{\mathrm{BH}} = \hat{H}^{(0)} + \hat{H}^{(1)}$$

$$\hat{H}^{(0)} = \sum_i \left[\frac{1}{2} U_0 \hat{n}_i (\hat{n}_i - 1) + \frac{1}{2} U_2 (\hat{\mathbf{S}}_i^2 - 2\hat{n}_i) - \mu \hat{n}_i - \eta \hat{S}_i^z \right]$$

$$\hat{H}^{(1)} = -J \sum_\alpha \sum_{<i,j>} \hat{a}_{i\alpha}^\dagger \hat{a}_{j\alpha}$$

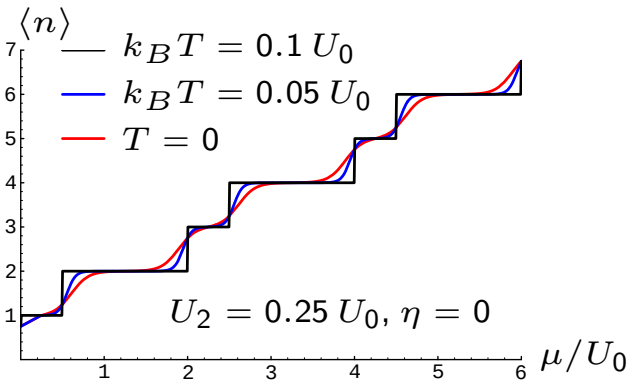
J: tunnel matrix element between nearest neighbors
$U_0 \propto a_0 + 2a_2$: spin independent interaction
$U_2 \propto a_0 - a_2$: spin dependent interaction
$\hat{n}_i$: particle number operator on site i
$\hat{\mathbf{S}}_i$: spin operators on site i with $[\hat{S}_j^\alpha, \hat{S}_k^\beta] = i\delta_{jk} \sum_\gamma \epsilon_{\alpha\beta\gamma} \hat{S}_j^\gamma$

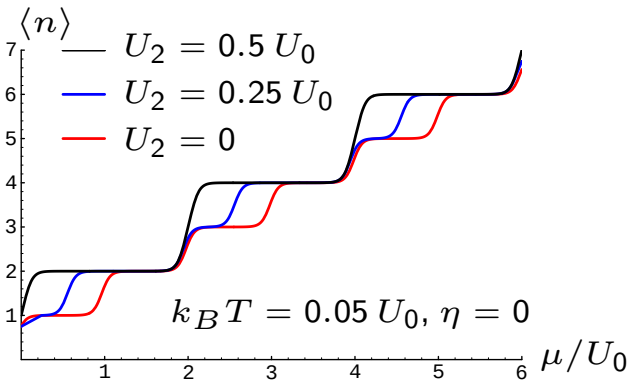
Thermal Properties of $J = 0$ System

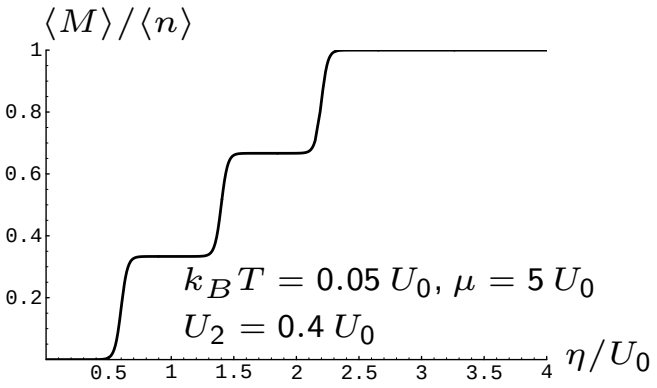
Hamiltonian site diagonal. Eigenstates characterized by particle number n, total spin S and z-component of spin m.

$$\hat{H}^{(0)}|S,m,n\rangle = N_S E^{(0)}_{S,m,n}|S,m,n\rangle$$

$$E^{(0)}_{S,m,n} = \frac{1}{2}U_0 n(n-1) + \frac{1}{2}U_2[S(S+1) - 2n] - \mu n - \eta m$$







Mean-Field Approximation and Landau Expansion

- Decoupling the hopping term:

$$\hat{a}^\dagger_{i\alpha}\hat{a}_{j\alpha} \approx \Psi_\alpha \hat{a}^\dagger_{i\alpha} + \Psi^*_\alpha \hat{a}_{j\alpha} - \Psi^*_\alpha \Psi_\alpha\,, \quad \Psi_\alpha = \langle \hat{a}_{i\alpha}\rangle\,, \quad \Psi^*_\alpha = \langle \hat{a}^\dagger_{i\alpha}\rangle$$

$$\hat{H}^{(1)}_{\mathsf{MF}} = -Jz \sum_i \sum_\alpha \left(\Psi_\alpha \hat{a}^\dagger_{i\alpha} + \Psi^*_\alpha \hat{a}_{i\alpha} - \Psi^*_\alpha \Psi_\alpha \right), \qquad z = 2D$$

- Perturbative expansion in $\hat{H}^{(1)}_{\mathsf{MF}}$ needs:

$$\hat{a}^\dagger_\alpha |S,m,n\rangle = M_{\alpha,S,m,n}|S{+}1, m{+}\alpha, n{+}1\rangle + N_{\alpha,S,m,n}|S{-}1, m{+}\alpha, n{+}1\rangle$$

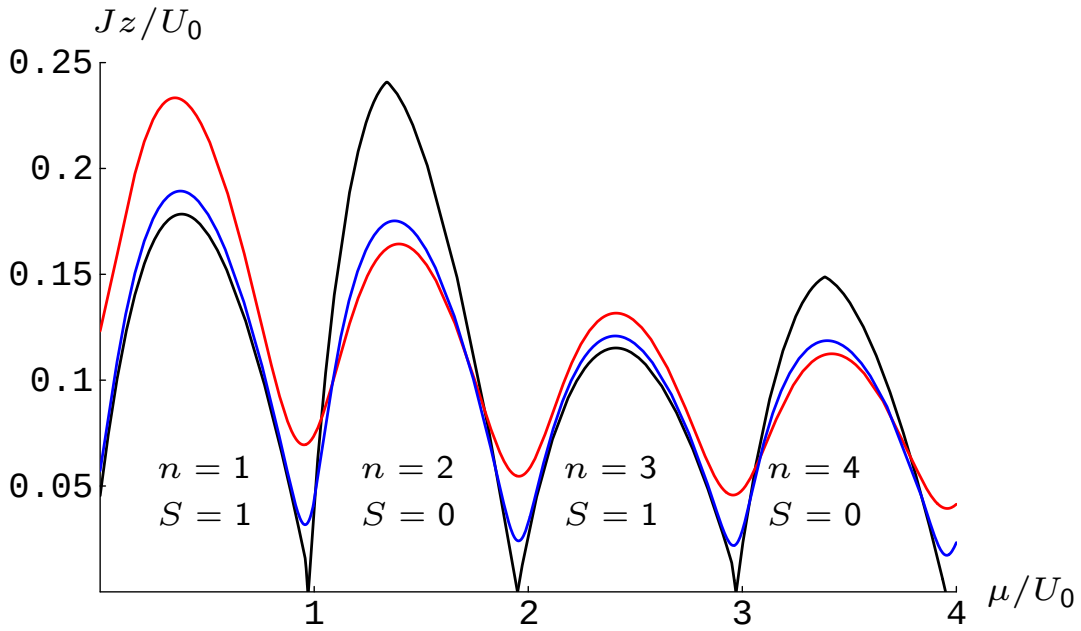
$$\hat{a}_\alpha |S,m,n\rangle = O_{\alpha,S,m,n}|S{+}1, m{-}\alpha, n{-}1\rangle + P_{\alpha,S,m,n}|S{-}1, m{-}\alpha, n{-}1\rangle$$

- Matrix elements M, N, O, P calculated by recursion relation
- Expanding the grand-canonical free energy:

$$\mathcal{F}(\mathbf{\Psi}^*, \mathbf{\Psi}) = -\,k_B T \log \mathsf{Tr}\left\{ e^{-(\hat{H}^{(0)} + \hat{H}^{(1)}_{\mathsf{MF}})/k_B T} \right\}$$

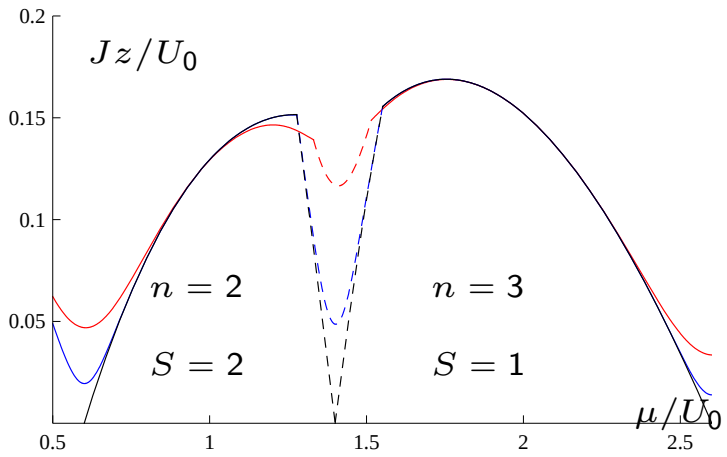
$$= -\,k_B T \log \mathcal{Z}^{(0)} + \sum_\alpha A^{(2)}_\alpha |\Psi_\alpha|^2 + O(\mathbf{\Psi}^4)$$

Phase Diagram



$U_2 = 0.04\,U_0$, $\eta = 0.05\,U_0$, $k_B T = 0.05\,U_0$ (red)

$k_B T = 0.09\,U_0$ (blue), $k_B T = 0$ (black)



Solid: $\Psi_1 \neq 0$, Dashed: $\Psi_{-1} \neq 0$

$U_2 = 0.5\,U_0$, $\eta = 0.09\,U_0$,

$k_B T = 0.05\,U_0$ (red), $k_B T = 0.02\,U_0$ (blue),

$k_B T = 0$ (black)

- Strong asymmetry between even and odd fillings for $T = 0$
- Thermal fluctuations lead to melting of singlet pairs and vanishing of asymmetry

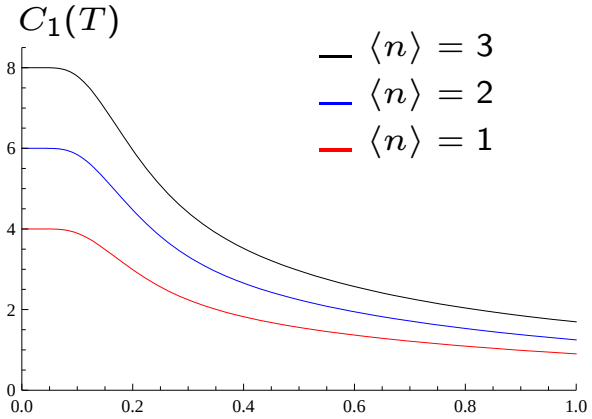
Visibility

- Extended definition for Spin-1 system

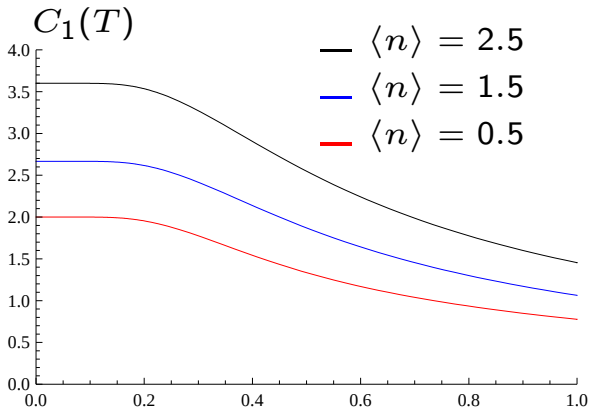
$$\mathcal{V}_\alpha = \frac{n_{\alpha\mathsf{max}} - n_{\alpha\mathsf{min}}}{n_{\alpha\mathsf{max}} + n_{\alpha\mathsf{min}}} \approx \frac{S_\alpha(\mathbf{k}_{\mathsf{max}}) - S_\alpha(\mathbf{k}_{\mathsf{min}})}{S_\alpha(\mathbf{k}_{\mathsf{max}}) + S_\alpha(\mathbf{k}_{\mathsf{min}})}$$

- First-order calculation for finite temperature:

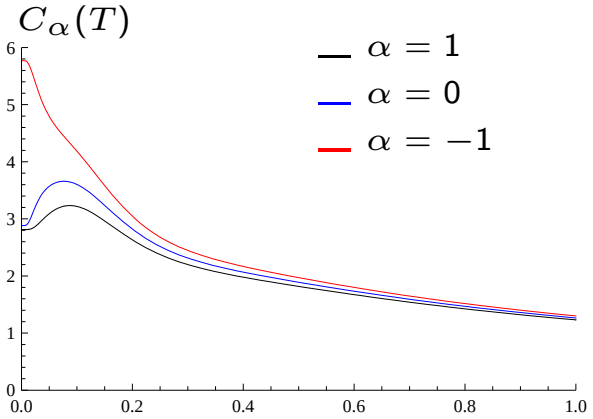
$$\mathcal{V}_\alpha(T) = 4zJ\left[1 - \cos(\sqrt{2}\pi)\right] C_\alpha(T) + O(J^2)$$



Fully polarized, integer filling



Fully polarized, fractional filling



$U_2 = 0.04\,U_0$, $\eta = 0.05\,U_0$, $\langle n \rangle = 1$

Summary

- Atoms in optical lattices provide unique model system for condensed matter physics
- Green's functions provide access to various properties of Bosons in optical lattices
- Diagramatic representation faciliates resummation
- Time-of-Flight pictures well explained especially in Mott phase
- One-loop corrected phase diagram in good agreement with Quantum Monte-Carlo data
- Effective masses of particle and hole excitations vanish for critical hopping parameter
- Spin-1 Bosons show richer phase diagram due to internal degrees of freedom

Outlook

- Scalar Model
  - Critical exponents of quantum phase transition
  - Green's function within superfluid phase: near phase boundary with Landau expansion, far away with Bogoliubov theory
  - Dynamic properties – Collapse and Revival
  - Four-point correlations – Hanbury-Brown-Twiss Effect

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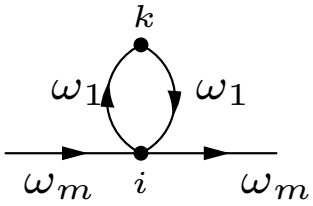
- Spinor Model
  - Different phases within superfluid
  - Quantum corrections
  - Order of phase transition

Calculation of One-Loop Diagram

$$2D\delta_{i,j}J^2G_1^{(2B)}(\omega) =$$



$$= \frac{2D\delta_{i,j}}{\mathcal{Z}^{(0)2}}\Bigg(\frac{1}{U^2}\sum_{n,k} e^{-\beta(E_n+E_k)}$$

$$\times\Bigg\{\frac{(k+1)(n-1)n\left[k^2+2(n-1)^2-\mu^2+2k(2-2n-\mu)\right]}{(k-n+1)^2(k-2n+\mu)(1-n+\mu)^2}$$

$$+\ 7\text{ more terms}\Bigg\} - C_1^{(0)}(\omega)^3$$

$$+\beta\Bigg\{\sum_{n,k}\left[\frac{(n+1)k}{k-n+1}+\frac{n(k+1)}{n-k+1}\right]\left[\frac{n+1}{n-\mu-i\omega}-\frac{n}{n-1-\mu-i\omega}\right]e^{-\beta(E_n+E_k)}$$

$$-C_1^{(0)}(\omega)\sum_{n,k}\left[\frac{(n+1)k}{k-n+1}+\frac{n(k+1)}{n-k+1}\right]e^{-\beta(E_n+E_k)}\Bigg\}\Bigg)$$

Calculation of Cumulants

- Generating functional:

$$C_0^{(0)}[j,j^*] = \log \left\langle \hat{T} \exp \left(\int_0^\beta d\tau j^*(\tau)\hat{a}(\tau) + j(\tau)\hat{a}^\dagger(\tau) \right) \right\rangle^{(0)}$$

- Cumulants calculated by functional derivatives:

$$C_n^{(0)}(\tau_1', \ldots, \tau_n' | \tau_1, \ldots, \tau_n) = \frac{\delta^{2n}}{\delta j(\tau_1') \ldots \delta j(\tau_n') \delta j^*(\tau_1) \ldots \delta j^*(\tau_n)} C_0^{(0)}[j,j^*] \Big|_{j=j^*=0}$$

$$C_1^{(0)}(\tau'|\tau) = \frac{1}{\mathcal{Z}^{(0)}} \sum_{n=0}^{\infty} \Big[\Theta(\tau-\tau')(n+1)\, e^{(E_n - E_{n+1})(\tau-\tau')} + \Theta(\tau'-\tau)\, n\, e^{(E_n - E_{n-1})(\tau'-\tau)} \Big]$$