



# Variational Resummation of Effective Potential in $\phi^4$ -Theory with Proper Goldstone Modes

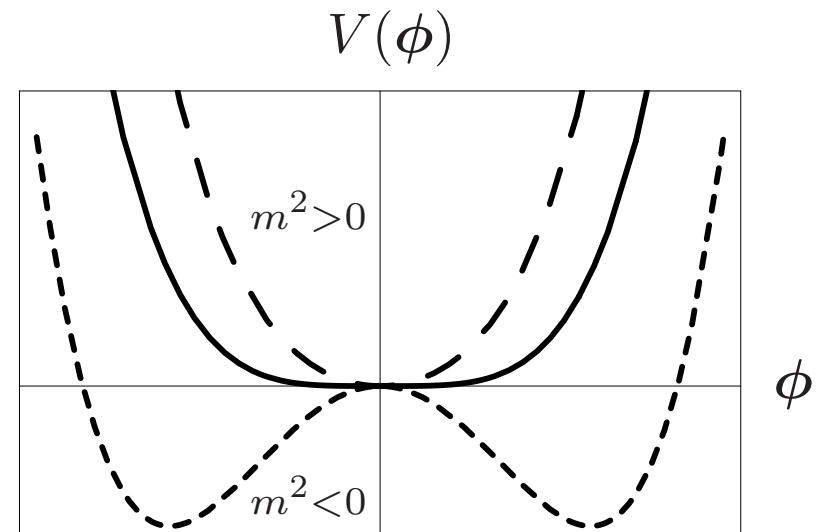
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$$\begin{aligned}V(\phi) &= \frac{1}{2}m^2\phi^2 + \frac{1}{24}g(\phi^2)^2 \\ \phi &= (\phi_1, \dots, \phi_N) \\ D &= 3\end{aligned}$$



# 1. Motivation

## Goldstone theorem:

*If the square mass becomes sufficiently negative, there is an ordered phase with a non-zero field expectation value and with Goldstone modes.*

**Goal:** Describing the phase transition with effective potential.

## Set of difficulties:

- Goldstone theorem must be preserved.
- Renormalization due to UV-divergencies.
- Resummation of divergent series.

## Previous treatments:

Hartree-approximation based on self-consistency conditions:

Goldstone theorem violated due to non-zero field expectation value.

Lenaghan and Rischke, J. Phys. G **26**, 431 (2000)

**Our approach:** variational perturbation theory (VPT)

## 2. Effective Potential for $N$ Real Fields

**Partition function:**

$$\mathcal{Z} = \int \mathcal{D}\phi e^{-E[\phi]}$$

**Energy:**

$$E[\phi] = \int d^Dx \left\{ -\frac{1}{2} \phi(\mathbf{x}) \Delta \phi(\mathbf{x}) + \frac{1}{2} m^2 \phi^2(\mathbf{x}) + \frac{1}{24} g [\phi^2(\mathbf{x})]^2 \right\}$$

**Background method:**

$$\phi(\mathbf{x}) = \Phi + \delta\phi(\mathbf{x})$$

**Effective potential:**

$$V_{\text{eff}}(\Phi) = -\frac{1}{\text{Vol}} \ln \mathcal{Z}(\Phi), \quad \Phi = |\Phi|$$

**Loop expansion in Feynman diagrams:**

$$V_{\text{eff}}(\Phi) = V(\Phi) + \frac{\hbar}{2} \text{O} + \frac{\hbar^2}{8} \text{O} \bullet \text{O} + \frac{\hbar^2}{12} \Phi \bullet \text{O} \bullet \Phi + \dots$$

$\hbar$ : artificial smallness parameter counting loop order

### 3. Sketch of Calculation

**Longitudinal and transversal masses:**

$$m_L^2(\Phi) = m^2 + \frac{g}{2} \Phi^2$$

$$m_T^2(\Phi) = m^2 + \frac{g}{6} \Phi^2$$

**Calculation of sunset term in  $D$  dimensions:**

Ford, Jack, and Jones, Nucl. Phys. B **387**, 373 (1992)

**UV-divergency for  $D = 3$ :** regularization with  $D = 3 + \varepsilon$

**Mass renormalization:**  $m_r^2 = m^2 + \frac{1}{\varepsilon} \frac{\hbar^2 g^2}{16\pi^2} + \dots$

**Note:**  $\phi^4$ -Theory is superrenormalizable in  $D = 3$  dimensions.

## 4. Necessity of Resummation

**Renormalized effective potential for  $D = 3$ :**

$$\begin{aligned}
 V_{\text{eff}}^{(r)}(\Phi) = & \frac{1}{2} m_r^2 \Phi^2 + \frac{g}{24} \Phi^4 - \frac{\hbar}{12\pi} \left\{ m_L^3(\Phi) + (N-1) m_T^3(\Phi) \right\} \\
 & + \frac{\hbar^2 g}{128\pi^2} \left\{ m_L^2(\Phi) + \frac{N^2-1}{3} m_T^2(\Phi) + \frac{2(N-1)}{3} m_L(\Phi) m_T(\Phi) \right\} \\
 & + \frac{\hbar^2 g^2 \Phi^2}{384\pi^2} \left\{ 2 \ln \frac{m_L(\Phi)}{\mu} - \ln \frac{4\pi}{9} - 1 + \gamma - \frac{N-1}{3} \left[ -\gamma + 1 + \ln 4\pi \right. \right. \\
 & \left. \left. + \ln \frac{m_L(\Phi)}{\mu} - \ln \left( \frac{2m_T(\Phi) - m_L(\Phi)}{\mu} \right) - 2 \ln \left( \frac{m_L(\Phi) + 2m_T(\Phi)}{\mu} \right) \right] \right\} + \dots
 \end{aligned}$$

**Loop expansion of background field:**

$$\left. \frac{\partial V_{\text{eff}}^{(r)}(\Phi)}{\partial \Phi} \right|_{\Phi=\Phi_{\text{ext}}} \stackrel{!}{=} 0 \quad \Rightarrow \quad \Phi_{\text{ext}} = \Phi_0 + \hbar \Phi_1 + \hbar^2 \Phi_2 + \dots$$

$\Phi_0$  and  $\Phi_1$  finite, but  $\Phi_2$  diverges due to  $m_T(\Phi_0) = 0 \Rightarrow$  resummation necessary

## 5. VPT with two Parameters

**Square-root substitution:**

$$m_L(\Phi) \longrightarrow \mathcal{M}_L \sqrt{1 + \hbar r_L},$$

$$r_L = \frac{m_L^2(\Phi) - \mathcal{M}_L^2}{\hbar \mathcal{M}_L^2},$$

$$m_T(\Phi) \longrightarrow \mathcal{M}_T \sqrt{1 + \hbar r_T}$$

$$r_T = \frac{m_T^2(\Phi) - \mathcal{M}_T^2}{\hbar \mathcal{M}_T^2}$$

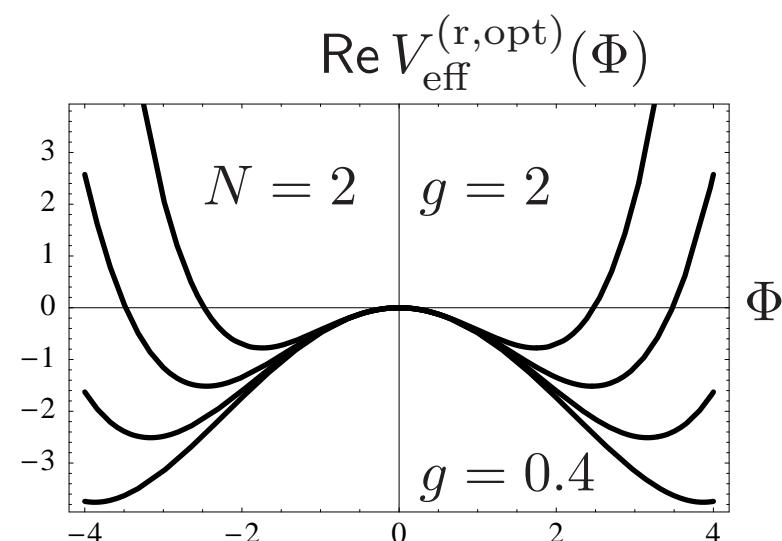
Procedure: Expansion in powers of  $\hbar$  and truncation at a certain order.

Consequence:  $V_{\text{eff}}^{(r)}(\Phi) \implies V_{\text{eff}}^{(r)}(\Phi, \mathcal{M}_L, \mathcal{M}_T)$

**Simultaneous optimization:**

$$\frac{\partial V_{\text{eff}}^{(r)}(\Phi, \mathcal{M}_L, \mathcal{M}_T)}{\partial \mathcal{M}_L} \stackrel{!}{=} 0$$

$$\frac{\partial V_{\text{eff}}^{(r)}(\Phi, \mathcal{M}_L, \mathcal{M}_T)}{\partial \mathcal{M}_T} \stackrel{!}{=} 0$$



## 6. Goldstone Theorem

### Goldstone Theorem:

If the square mass becomes sufficiently negative, there is an ordered phase with a non-zero field expectation value and with Goldstone modes.

### Verification of Goldstone Theorem:

Rotationally symmetric renormalized effective potential  $V_{\text{eff}}^{(r)}(\Phi)$ .

Transversal projector:  $P_{ij}^T = \delta_{ij} - \frac{\Phi_i \Phi_j}{\Phi^2}$

Transversal masses: 
$$P_{ij}^T \frac{\partial^2 V_{\text{eff}}^{(r)}(\Phi)}{\partial \Phi_i \partial \Phi_j} \Bigg|_{\Phi=\Phi_{\text{ext}}} = (N-1) \frac{\partial V_{\text{eff}}^{(r)}(\Phi)}{\partial \Phi} \Bigg|_{\Phi=\Phi_{\text{ext}}} = 0$$

Calculation valid after renormalization and resummation.

## 7. Landau Expansion

**Universal ansatz:**  $V_{\text{eff}}^{(\text{r})}(\Phi) = C + \frac{1}{2} m_{\text{ph}}^2 \Phi^2 + \frac{1}{24} g_{\text{ph}} \Phi^4 + \dots$

**Results:**

stability:  $g_{\text{ph}} = \left. \frac{\partial^4 V_{\text{eff}}^{(\text{r})}(\Phi)}{\partial \Phi^4} \right|_{\Phi=0} > 0$

phase transition:  $m_{\text{ph}}^2 = \left. \frac{\partial^2 V_{\text{eff}}^{(\text{r})}(\Phi)}{\partial \Phi^2} \right|_{\Phi=0} \stackrel{!}{=} 0 \implies m_{\text{r,c}}^2 = 0.00570 \quad (N=2, g=1)$

- Physical interpretation:  $m_{\text{r,c}}^2 \sim T_c - T_c^{(0)} > 0 \implies T_c > T_c^{(0)}$   
Positive shift of critical temperature agrees qualitatively with result for homogeneous Bose gas

Kleinert, Mod. Phys. Lett. B **17**, 1011 (2003), Kastening, Phys. Rev. A **69**, 043613 (2004)

Arnold and Moore, Phys. Rev. Lett. **87**, 120401 (2001),

Kashurnikov, Prokof'ev, and Svistunov, Phys. Rev. Lett. **87**, 120402 (2001)

- Without sunset diagram no second-order phase transition.

## 8. Large- $N$ Limit

- Hubbard-Stratonovich transformation focuses on **Hartree channel**: valid approximation in the limit  $N \rightarrow \infty$
- VPT  $\xrightarrow{N \rightarrow \infty}$  **Hubbard-Stratonovich transformation**: transversal mass  $\iff$  Hubbard-Stratonovich background field

## 9. Summary

- **Conclusion:**  
VPT is a consistent generalization of the Hubbard-Stratonovich transformation to an arbitrary number  $N$  of fields which does not violate the Goldstone theorem  
Kleinert, Ann. Phys. (New York) **266**, 135 (1998)
- **Perspective:**  
Extension to finite temperatures (Bose-Einstein Condensation)