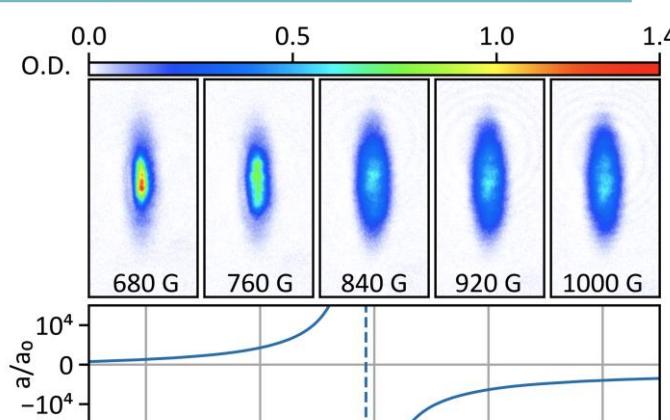


Thermometry for trapped fermionic atoms in the BCS limit

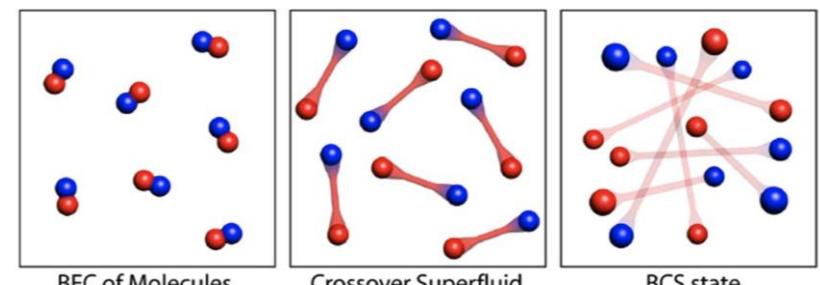
S. Yong, S. Barbosa, J. Koch, A. Widera and A. Pelster

Contents

- Introduction
- Mean-field approach
- Experiment
- Conclusion & Outlook



B. Gänger, Phieler, B. J., Nagler, & A. Widera, Rev. Sci. Instr. (2018)



W. Ketterle, & M. W. Zwierlein, La Rivista del Nuovo Cimento (2008)



Introduction: History of Many-Body theory

- Functional integral with Grassmann fields
- Hubbard-Stratonovich transformation:
Effective action of bosonic pairing field (*)
- Saddle point approximation: BCS-pairing below T_{pair}
- Gaussian fluctuation of pairing field around saddle point: BCS-BEC crossover at T_c
- Technical problems:
 1. Above T_{pair} : ideal Fermi gas
 2. Gaussian fluctuation is numerically demanding

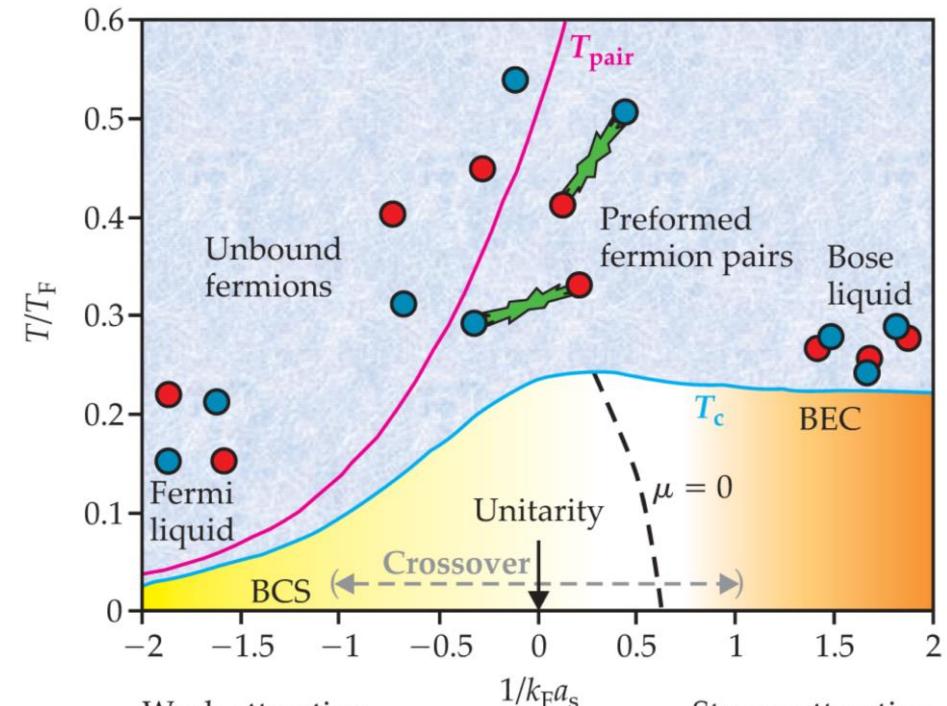


Fig.: BCS-BEC crossover phase diagram (**), originally published in (*).

Mean-field approach: HFB-approximation

- Second quantised Hamiltonian
- Two-body contact interaction $g \propto a_s$
- Quartic interaction → quadratic operators with mean-field expectation values
- Classification of interaction channels:

HFB-approximation

$$\hat{\psi}_\uparrow^\dagger \hat{\psi}_\downarrow^\dagger \hat{\psi}_\downarrow \hat{\psi}_\uparrow \approx + \underbrace{\langle \hat{\psi}_\uparrow^\dagger \hat{\psi}_\uparrow \rangle \hat{\psi}_\downarrow^\dagger \hat{\psi}_\downarrow + \hat{\psi}_\uparrow^\dagger \hat{\psi}_\uparrow \langle \hat{\psi}_\downarrow^\dagger \hat{\psi}_\downarrow \rangle - \langle \hat{\psi}_\uparrow^\dagger \hat{\psi}_\uparrow \rangle \langle \hat{\psi}_\downarrow^\dagger \hat{\psi}_\downarrow \rangle}_{\text{Hartree channel}}$$

$$- \underbrace{\langle \hat{\psi}_\uparrow^\dagger \hat{\psi}_\downarrow \rangle \hat{\psi}_\downarrow^\dagger \hat{\psi}_\uparrow - \hat{\psi}_\uparrow^\dagger \hat{\psi}_\downarrow \langle \hat{\psi}_\downarrow^\dagger \hat{\psi}_\uparrow \rangle + \langle \hat{\psi}_\uparrow^\dagger \hat{\psi}_\downarrow \rangle \langle \hat{\psi}_\downarrow^\dagger \hat{\psi}_\uparrow \rangle}_{\text{Fock channel}}$$

Bogoliubov channel

$$\text{MF-BCS theory (*)} \longrightarrow + \underbrace{\langle \hat{\psi}_\uparrow^\dagger \hat{\psi}_\downarrow^\dagger \rangle \hat{\psi}_\downarrow \hat{\psi}_\uparrow + \hat{\psi}_\uparrow^\dagger \hat{\psi}_\downarrow \langle \hat{\psi}_\downarrow \hat{\psi}_\uparrow \rangle - \langle \hat{\psi}_\uparrow^\dagger \hat{\psi}_\downarrow^\dagger \rangle \langle \hat{\psi}_\downarrow \hat{\psi}_\uparrow \rangle}_{\text{Bogoliubov channel}}$$

$$\hat{H} = \int d^3x \left\{ \sum_{\sigma=\uparrow,\downarrow} \hat{\psi}_\sigma^\dagger(\vec{x}) \left[-\frac{\hbar^2}{2M} \nabla^2 - \mu \right] \hat{\psi}_\sigma(\vec{x}) + g \hat{\psi}_\uparrow^\dagger(\vec{x}) \hat{\psi}_\downarrow^\dagger(\vec{x}) \hat{\psi}_\downarrow(\vec{x}) \hat{\psi}_\uparrow(\vec{x}) \right\}$$

Mean-field HFB-approximation

$$\hat{H}_{\text{MF}} = \int d^3x \left\{ \sum_{\sigma=\uparrow,\downarrow} \hat{\psi}_\sigma^\dagger(\vec{x}) \left[-\frac{\hbar^2}{2M} \nabla^2 - \mu \right] \hat{\psi}_\sigma(\vec{x}) + \Gamma_\downarrow^{\text{H}} \hat{\psi}_\uparrow^\dagger(\vec{x}) \hat{\psi}_\uparrow(\vec{x}) + \Gamma_\uparrow^{\text{H}} \hat{\psi}_\downarrow^\dagger(\vec{x}) \hat{\psi}_\downarrow(\vec{x}) - \frac{\Gamma_\uparrow^{\text{H}} \Gamma_\downarrow^{\text{H}}}{g} \right. \\ \left. + \Gamma_\downarrow^{\text{F}} \hat{\psi}_\uparrow^\dagger(\vec{x}) \hat{\psi}_\downarrow(\vec{x}) - \Gamma_\uparrow^{\text{F}} \hat{\psi}_\downarrow^\dagger(\vec{x}) \hat{\psi}_\uparrow(\vec{x}) + \frac{\Gamma_{\uparrow\downarrow}^{\text{F}} \Gamma_{\downarrow\uparrow}^{\text{F}}}{g} \right. \\ \left. + \Delta^* \hat{\psi}_\downarrow(\vec{x}) \hat{\psi}_\uparrow(\vec{x}) + \Delta \hat{\psi}_\uparrow(\vec{x}) \hat{\psi}_\downarrow(\vec{x}) - \frac{\Delta^* \Delta}{g} \right\}$$

$\Gamma_\uparrow^{\text{H}} := g \langle \hat{\psi}_\uparrow^\dagger \hat{\psi}_\uparrow \rangle; \Gamma_\downarrow^{\text{H}} := g \langle \hat{\psi}_\downarrow^\dagger \hat{\psi}_\downarrow \rangle \quad (\text{Hartree channel})$
 $\Gamma_{\uparrow\downarrow}^{\text{F}} := g \langle \hat{\psi}_\uparrow^\dagger \hat{\psi}_\downarrow \rangle; \Gamma_{\downarrow\uparrow}^{\text{F}} := g \langle \hat{\psi}_\downarrow^\dagger \hat{\psi}_\uparrow \rangle \quad (\text{Fock channel})$
 $\Delta := g \langle \hat{\psi}_\downarrow \hat{\psi}_\uparrow \rangle; \Delta^* := g \langle \hat{\psi}_\uparrow^\dagger \hat{\psi}_\downarrow^\dagger \rangle \quad (\text{Bogoliubov channel})$

Mean-field approach: Different models

- Fourier transformation for diagonalisation
- Choice of interaction channels leads to different models:

-BCS-Bogoliubov model (*):

- Condensation below T_{pair}

-Hartree model:

- Normal fluid with Hartree-interaction
- Fock Contribution vanishes due to contact interaction

-Hartree-Bogoliubov model:

- Condensation + Hartree-interaction

$$\hat{H}_{\text{MF}} = \sum_{\vec{k}} \left\{ \left[\frac{\hbar^2 \vec{k}^2}{2M} - \mu + \Gamma_{\downarrow}^{\text{H}} \right] \hat{a}_{\vec{k},\uparrow}^\dagger \hat{a}_{\vec{k},\uparrow} + \left[\frac{\hbar^2 \vec{k}^2}{2M} - \mu + \Gamma_{\uparrow}^{\text{H}} \right] \hat{a}_{\vec{k},\downarrow}^\dagger \hat{a}_{\vec{k},\downarrow} \right. \\ - \Gamma_{\uparrow\downarrow}^{\text{F}} \hat{a}_{\vec{k},\downarrow}^\dagger \hat{a}_{\vec{k},\uparrow} - \Gamma_{\downarrow\uparrow}^{\text{F}} \hat{a}_{\vec{k},\uparrow}^\dagger \hat{a}_{\vec{k},\downarrow} \\ \left. + \Delta^* \hat{a}_{-\vec{k},\downarrow} \hat{a}_{\vec{k},\uparrow} + \Delta \hat{a}_{\vec{k},\uparrow}^\dagger \hat{a}_{-\vec{k},\downarrow} \right\} \\ - V \frac{\Gamma_{\uparrow}^{\text{H}} \Gamma_{\downarrow}^{\text{H}}}{g} + V \frac{\Gamma_{\uparrow\downarrow}^{\text{F}} \Gamma_{\downarrow\uparrow}^{\text{F}}}{g} - V \frac{\Delta^* \Delta}{g}$$

$$\Gamma_{\uparrow}^{\text{H}} := g \langle \hat{\psi}_\uparrow^\dagger \hat{\psi}_\uparrow \rangle; \Gamma_{\downarrow}^{\text{H}} := g \langle \hat{\psi}_\downarrow^\dagger \hat{\psi}_\downarrow \rangle \quad (\text{Hartree channel})$$

$$\Gamma_{\uparrow\downarrow}^{\text{F}} := g \langle \hat{\psi}_\uparrow^\dagger \hat{\psi}_\downarrow \rangle; \Gamma_{\downarrow\uparrow}^{\text{F}} := g \langle \hat{\psi}_\downarrow^\dagger \hat{\psi}_\uparrow \rangle \quad (\text{Fock channel})$$

$$\Delta := g \langle \hat{\psi}_\downarrow \hat{\psi}_\uparrow \rangle; \Delta^* := g \langle \hat{\psi}_\uparrow^\dagger \hat{\psi}_\downarrow^\dagger \rangle \quad (\text{Bogoliubov channel})$$

Mean-field approach: BCS-Bogoliubov model

- Bogoliubov channel for condensation

$$\hat{H}_B = \sum_{\vec{k}} \epsilon_B(\vec{k}) (\hat{\alpha}_{\vec{k}}^\dagger \hat{\alpha}_{\vec{k}} + \hat{\beta}_{\vec{k}}^\dagger \hat{\beta}_{\vec{k}}) - \sum_{\vec{k}} \left\{ \epsilon_B(\vec{k}) - \left(\frac{\hbar^2 \vec{k}^2}{2M} - \mu \right) \right\} - V \frac{|\Delta|^2}{g}$$

- Diagonalisation: Bogoliubov transformation

$$\hat{\alpha}_{\vec{k}} = u_{\vec{k}} \hat{a}_{\vec{k},\uparrow} - v_{\vec{k}} \hat{a}_{-\vec{k},\downarrow}^\dagger,$$

$$\hat{\beta}_{\vec{k}} = u_{\vec{k}} \hat{a}_{-\vec{k},\downarrow} + v_{\vec{k}} \hat{a}_{\vec{k},\uparrow}^\dagger,$$

- Thermodynamic potential $\mathcal{F} = -\beta^{-1} \log \text{Tr} e^{-\beta \hat{H}}$

→ Order parameter equation: $\partial_{\Delta^*} \mathcal{F} = 0$

→ Particle number equation: $-\partial_\mu \mathcal{F} = N$

- Equivalent to saddle point approximation of functional integral formulation

renormalised
BCS-order parameter equation

$$\frac{1}{g_r} = \frac{1}{2V} \sum_{\vec{k}} \left\{ \frac{2M}{\hbar^2 \vec{k}^2} - \frac{1}{\epsilon_B(\vec{k})} \tanh \left(\frac{\beta \epsilon_B(\vec{k})}{2} \right) \right\}$$

BCS-density equation

$$n = \frac{1}{V} \sum_{\vec{k}} \left\{ 1 - \frac{\frac{\hbar^2 \vec{k}^2}{2M} - \mu}{\epsilon_B(\vec{k})} \tanh \left(\frac{\beta \epsilon_B(\vec{k})}{2} \right) \right\}$$

$$\epsilon_B(\vec{k}) := \sqrt{\left(\frac{\hbar^2 \vec{k}^2}{2M} - \mu \right)^2 + \Delta^2}$$

Mean-field approach: Hartree model

- Hartree-interaction between fermions
- Thermodynamic potential $\mathcal{F} = -\beta^{-1} \log \text{Tr} e^{-\beta \hat{H}}$
- Particle number equation: $-\partial_\mu \mathcal{F} = N$
- Hartree equations:

$$\frac{\partial \mathcal{F}}{\partial \Gamma_{\downarrow}^H} = 0 \Rightarrow \frac{\Gamma_{\downarrow}^H}{g} = \frac{1}{V} \sum_{\vec{k}} \frac{1}{e^{-\beta \left(\frac{\hbar^2 \vec{k}^2}{2M} - \mu + \Gamma_{\downarrow}^H \right)} + 1}$$

$$\frac{\partial \mathcal{F}}{\partial \Gamma_{\uparrow}^H} = 0 \Rightarrow \frac{\Gamma_{\uparrow}^H}{g} = \frac{1}{V} \sum_{\vec{k}} \frac{1}{e^{-\beta \left(\frac{\hbar^2 \vec{k}^2}{2M} - \mu + \Gamma_{\uparrow}^H \right)} + 1}$$

$$\Gamma_{\downarrow}^H = \Gamma_{\uparrow}^H = \frac{1}{2} g n \leftarrow \Gamma$$

$$\hat{H}_H = \sum_{\vec{k}} \left\{ \left[\frac{\hbar^2 \vec{k}^2}{2M} - \mu + \Gamma_{\downarrow}^H \right] \hat{a}_{\vec{k},\uparrow}^\dagger \hat{a}_{\vec{k},\uparrow} + \left[\frac{\hbar^2 \vec{k}^2}{2M} - \mu + \Gamma_{\uparrow}^H \right] \hat{a}_{\vec{k},\downarrow}^\dagger \hat{a}_{\vec{k},\downarrow} \right\} - V \frac{\Gamma_{\downarrow}^H \Gamma_{\uparrow}^H}{g}$$

Coincidence with result from effective single-body approximation approach of Hartree-Fock theory (*), $\Gamma^F = 0$ for contact interaction

Hartree-density equation

$$n = -\frac{2}{\lambda^3} \text{Li}_{3/2} \left(-e^{\beta(\mu - \frac{1}{2}gn)} \right)$$

$$\lambda = \sqrt{\frac{2\pi\hbar^2}{MK_B T}}$$

$$\text{Li}_s(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^s}$$

Mean-field approach: Hartree-Bogoliubov model

- Taking Hartree- and Bogoliubov channels:
Interaction and condensation
- Extremalisation of thermodynamic potential:
 $\partial_{\Delta^*} \mathcal{F} = 0$: Order parameter equation
 $\partial_{\Gamma^H} \mathcal{F} = 0$: Hartree equation
 $-\partial_\mu \mathcal{F} = N$: Particle number equation
- Results:
- $T_{\text{pair}}^{(\text{B})} < T_{\text{pair}}^{(\text{HB})}$
- $\Delta^{(\text{B})} < \Delta^{(\text{HB})}$

$$\hat{H}_{\text{HB}} = \sum_{\vec{k}} \epsilon_{\text{HB}}(\vec{k}) (\hat{\alpha}_{\vec{k}}^\dagger \hat{\alpha}_{\vec{k}} + \hat{\beta}_{\vec{k}}^\dagger \hat{\beta}_{\vec{k}}) - \sum_{\vec{k}} \left\{ \epsilon_{\text{HB}}(\vec{k}) - \left(\frac{\hbar^2 \vec{k}^2}{2M} - \mu + \Gamma \right) \right\} - V \frac{\Gamma^2}{g} - V \frac{|\Delta|^2}{g}$$

renormalised
HB-order parameter equation

$$\frac{1}{g_r} = \frac{1}{2V} \sum_{\vec{k}} \left\{ \frac{2M}{\hbar^2 \vec{k}^2} - \frac{1}{\epsilon_{\text{HB}}(\vec{k})} \tanh \left(\frac{\beta \epsilon_{\text{HB}}(\vec{k})}{2} \right) \right\}$$

HB-density equation

$$n = \frac{1}{V} \sum_{\vec{k}} \left\{ 1 - \frac{\frac{\hbar^2 \vec{k}^2}{2M} - \mu + \Gamma}{\epsilon_{\text{HB}}(\vec{k})} \tanh \left(\frac{\beta \epsilon_{\text{HB}}(\vec{k})}{2} \right) \right\}$$

$$\epsilon_{\text{HB}}(\vec{k}) := \sqrt{\left(\frac{\hbar^2 \vec{k}^2}{2M} - \mu + \Gamma \right)^2 + |\Delta|^2}$$



Mean-field approach: Trapped Fermi gas with LDA

- Harmonic trapping potential:

$$V(\vec{x}) = \frac{M}{2} \sum_{i=1}^3 \omega_i^2 x_i^2$$

- Local Density Approximation (LDA)

$$\mu \rightarrow \mu - V(\vec{x}) \Rightarrow n \rightarrow n(\vec{x})$$

- Experimental parameters and measured quantities:

$$1/k_F a_s, (\omega_1, \omega_2, \omega_3), N$$

- Free fitting parameters: T, μ

\Rightarrow Choose T and fit μ for $N = \int d^3x n(\vec{x})$

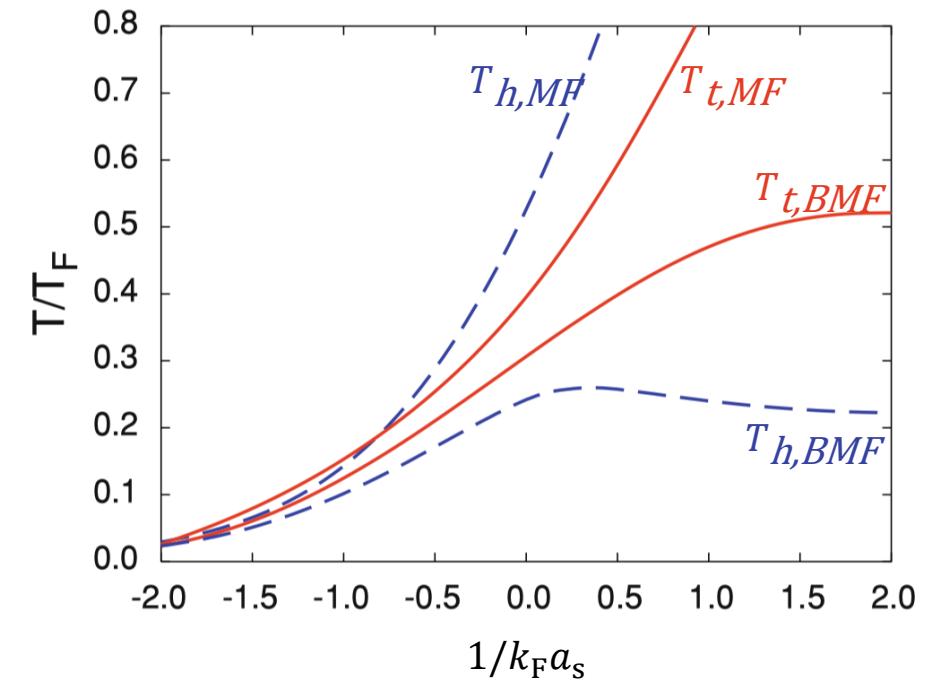
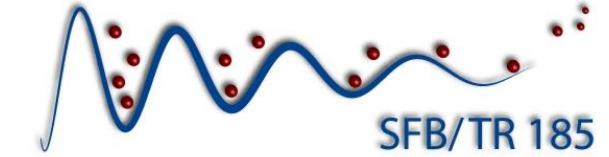


Fig.: Critical temperatures of homogenous(h)(*)- and trapped(t)(**) gas. The plot is taken from (**).

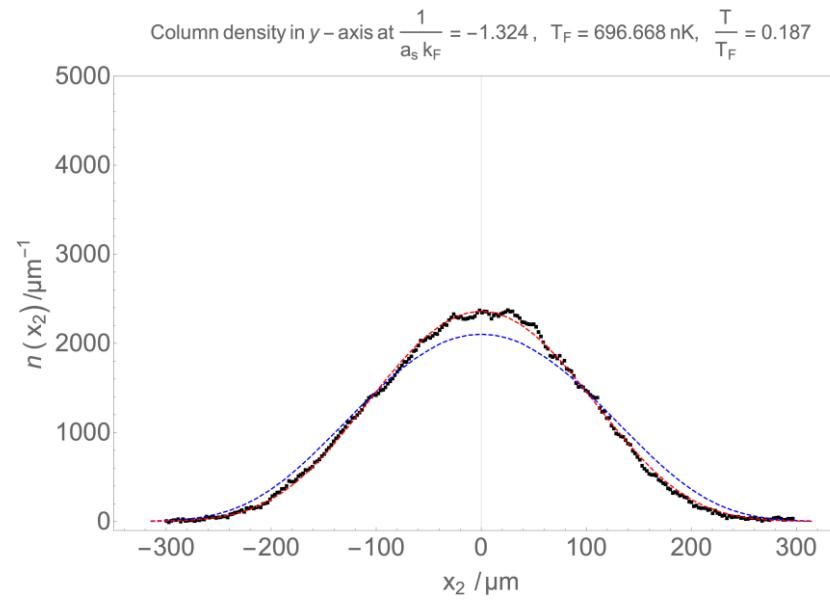
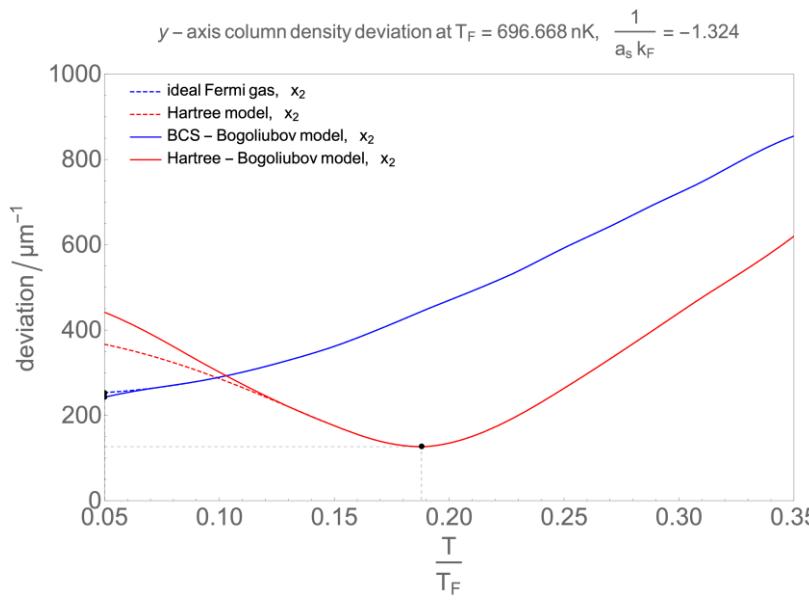
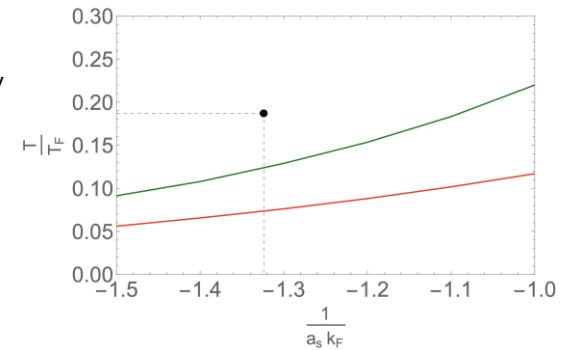
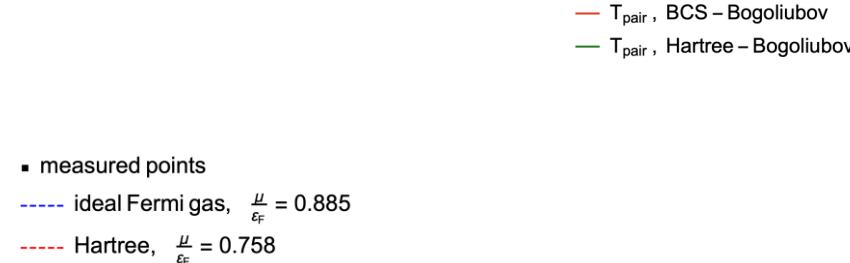
$$\begin{aligned}
 k_F^{(hom)} &= (3\pi^2 n)^{1/3} \\
 k_F^{(trap)} &= 2^{1/2} 3^{1/6} \sqrt{\frac{M(\omega_1 \omega_2 \omega_3)^{1/3}}{\hbar}} N^{1/6}
 \end{aligned}$$

RPTU



Experiment: Normal fluid phase

$$1/k_F a_s = -1.324 \rightarrow T = 0.187 T_F$$



Experimental data due to Widera Group

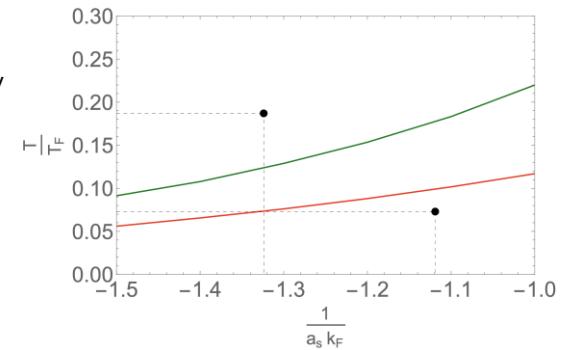
Experiment: Superfluid phase



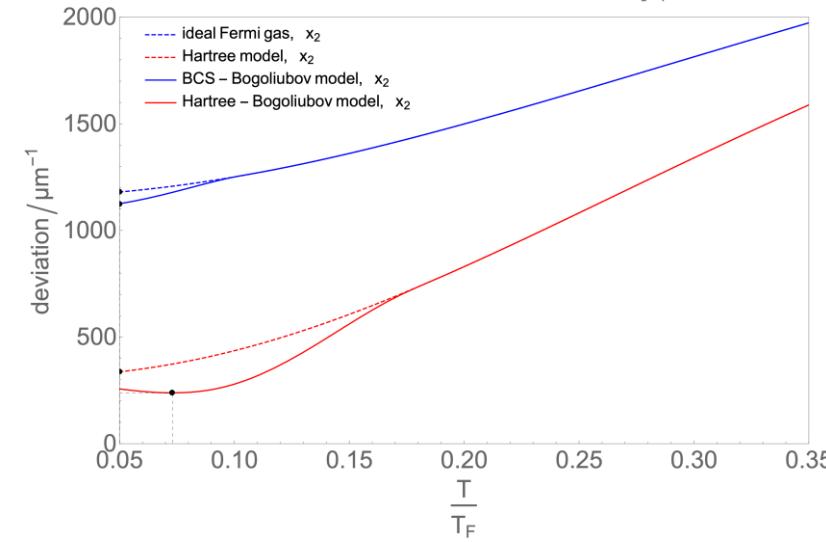
$$1/k_F a_s = -1.119 \rightarrow T = 0.073 T_F$$

- measured points
- - - ideal Fermi gas, $\frac{\mu}{\epsilon_F} = 0.982$
- - - BCS – Bogoliubov, $\frac{\mu}{\epsilon_F} = 0.981$
- - - Hartree, $\frac{\mu}{\epsilon_F} = 0.8$
- - - Hartree – Bogoliubov, $\frac{\mu}{\epsilon_F} = 0.782$

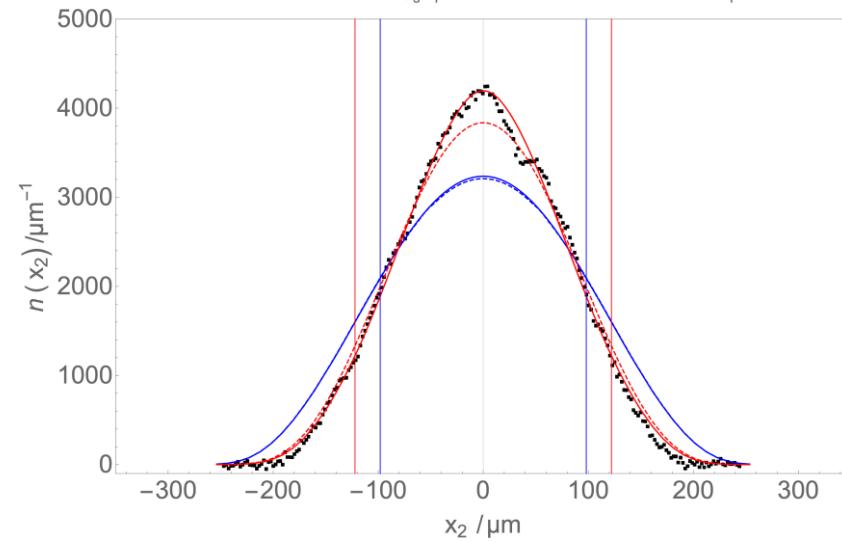
— T_{pair} , BCS – Bogoliubov
— T_{pair} , Hartree – Bogoliubov



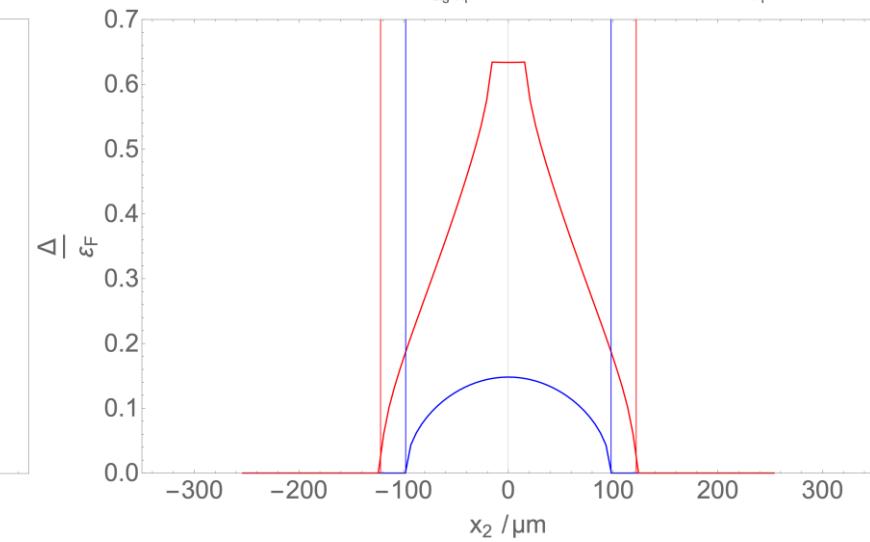
y – axis column density deviation at $T_F = 565.576$ nK, $\frac{1}{a_s k_F} = -1.119$



Column density in y – axis at $\frac{1}{a_s k_F} = -1.119$, $T_F = 565.576$ nK, $\frac{T}{T_F} = 0.073$

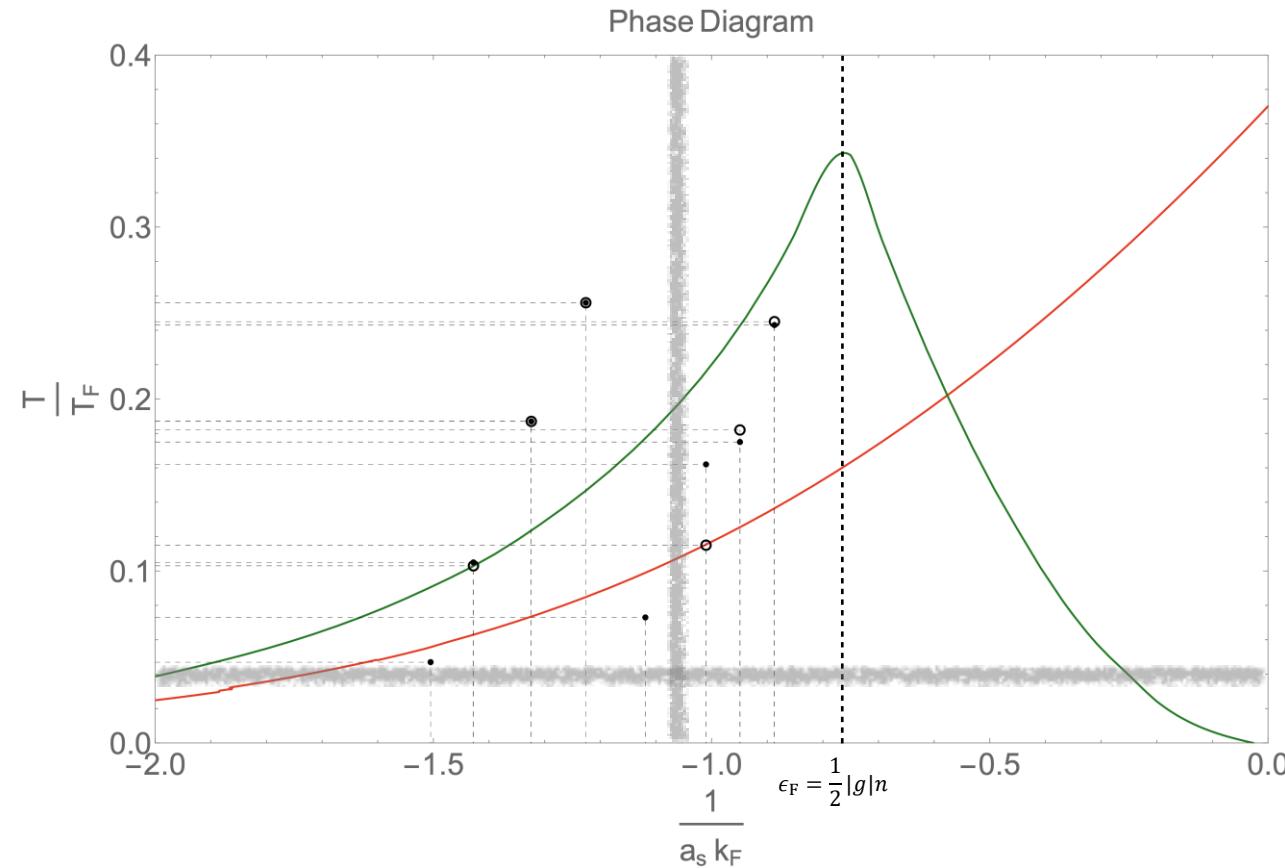


Order parameter in y – axis at $\frac{1}{a_s k_F} = -1.119$, $T_F = 565.576$ nK, $\frac{T}{T_F} = 0.073$



Experiment: Overview

- T_{pair} , BCS – Bogoliubov (*)
- T_{pair} , Hartree – Bogoliubov
- Thermometry by Hartree theory
- Thermometry by HB theory



Experimental data due to Widera Group

RPTU

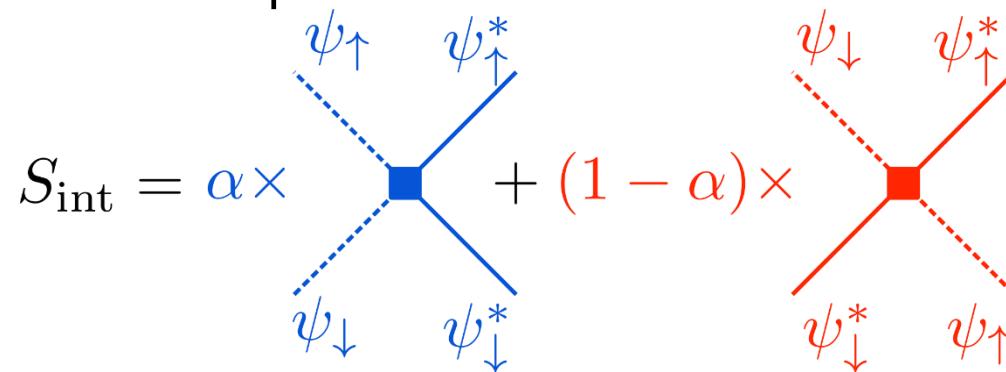
Conclusion and Outlook

- Interaction effect is important in both normal- and superfluid phase
→ Hartree-interaction should be included
- Direct thermometry of trapped interacting Fermi gas in BCS-limit achieved
- Application: Machine learning for phase diagram of interacting Fermi gas (*)
→ Supervised learning method with HFB theory
- Problem: Closer to unitarity fluctuations get so important that MF theories fail
→ Remedy: Beyond mean-field HFB theory, or extend variationally HFB theory

Variational multi-channel mean field theory with finite range

(N. Kaschewski, A. Pelster and C.A.R Sá de Melo)

- Variational approach to action S
- Distribute action between pairing (**Bogoliubov**) and direct (**Hartree**) channel via variational parameter α :



- Introduce additional saddle point conditions for the Hartree energy and the variational parameter
- Renormalization of effective pairing interaction
 - Dependence on interaction range and density

Preliminary Result: Homogeneous ${}^6\text{Li}$ gas

