Two Intriguing Examples for Topological Effects in Ultra-Cold Atoms

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SFB/Transregio 49
Frankfurt – Kaiserslautern - Mainz
Condensed matter systems with variable many-body interactions

SFB/TR 185

VolkswagenStiftung

1. Ground-State Properties of Anyons in a One-Dimensional Lattice

Tang, Eggert, and Pelster, NJP **17**, 123016 (2015)

2. Quantum Domain Walls Induce Incommensurate Supersolid Phase on the Anisotropic Triangular Lattice

Zhang, Hu, Pelster, and Eggert, PRL **117**, 193210 (2016)

3. Outlook

1.1 Identical Quantum Particles

- $D = 3$: **Bosons and fermions**
- $D = 2$: **Anyons obeying fractional statistics**
  - **Exchange of two anyons:** phase factor $e^{i\theta}$ with $0 \leq \theta \leq \pi$
  - **Example:** electrons in fractional quantum Hall effect
- **Interpolation between Bose-Einstein and Fermi-Dirac statistics:**
  - **E.g.** Gentile (1940), Haldane (1991), Polychronakos (1996), ...
  - **Virial expansion:** those fractional statistics disagree with anyons

    Khare, *Fractional Statistics and Quantum Theory* (World Scientific, 2005)

  $\Longrightarrow$ **Anyon statistics not yet known**
- $D = 1$: **Possible realizations of anyons with quantum gases**
  - **Photon-assisted tunneling:** Keilmann et al., NC **2**, 361 (2011)
  - **Raman scheme:** Greschner and Santos, PRL **115**, 053002 (2015)
  - **Lattice-shaking-induced resonant tunneling:**

    Sträter, Srivastava, and Eckardt, PRL **117**, 205303 (2016)

1.2 Anyon Model in 1D

- **Anyon-Hubbard Hamilton:**

$$\hat{H}^a = -J \sum_{j=1}^{L} (\hat{a}_j^\dagger \hat{a}_{j+1} + \text{h.c.}) + \frac{U}{2} \sum_{j=1}^{L} \hat{n}_j (\hat{n}_j - 1)$$

$$\begin{aligned} \hat{a}_j \hat{a}_k^\dagger - e^{-\mathrm{i}\theta \mathrm{sgn}(j-k)} \hat{a}_k^\dagger \hat{a}_j &= \delta_{jk} \\ \hat{a}_j \hat{a}_k - e^{\mathrm{i}\theta \mathrm{sgn}(j-k)} \hat{a}_k \hat{a}_j &= 0 \end{aligned}$$

$\theta = 0$**: bosons,** $\theta = \pi$**: pseudofermions**

- **Jordan-Wigner transformation:** $\hat{a}_j = \hat{b}_j \exp\left(\mathrm{i}\theta \sum_{i=1}^{j-1} \hat{n}_i\right)$

- **Occupation-dependent hopping Bose-Hubbard model:**

$$\hat{H}^b = -J \sum_{j=1}^{L} (\hat{b}_j^\dagger \hat{b}_{j+1} e^{\mathrm{i}\theta \hat{n}_j} + \text{h.c.}) + \frac{U}{2} \sum_{j=1}^{L} \hat{n}_j (\hat{n}_j - 1)$$

$$\begin{aligned} \hat{b}_j \hat{b}_k^\dagger - \hat{b}_k^\dagger \hat{b}_j &= \delta_{jk} \\ \hat{b}_j \hat{b}_k - \hat{b}_k \hat{b}_j &= 0 \end{aligned}$$

Keilmann et al., NC **2**, 361 (2011)

1.3 Gutzwiller Mean-Field Approximation

- **Product ansatz:** $$|G\rangle = \prod_j \left(\sum_{n=0}^{n_{\max}} f_n^{(j)} |n\rangle \right)$$

- **Normalization:** $$\sum_{n=0}^{n_{\max}} |f_n^{(j)}|^2 = 1$$

- **Energy:** $$E\left(\left\{ f_n^{(j)} \right\}\right) = \langle G|\hat{H}^b|G\rangle$$

- **Classical GW approach:** $$f_n^{(j)} = f_n$$

- **Modified GW approach 1:** $$f_n^{(j)} = F_n \, e^{i\alpha_n^{(j)}}$$
  Tang, Eggert, and Pelster, NJP **17**, 123016 (2015)

- **Modified GW approach 2:** $$f_n^{(j)} = F_n \, e^{i(\alpha_n + j\beta_n)}$$
  Bonkhoff, diploma thesis, TU Kaiserslautern (2016)

- **Energy extremization yields Gutzwiller amplitudes and phases**

1.4 Quasi-Momentum Distributions

- **Bosonic version:**

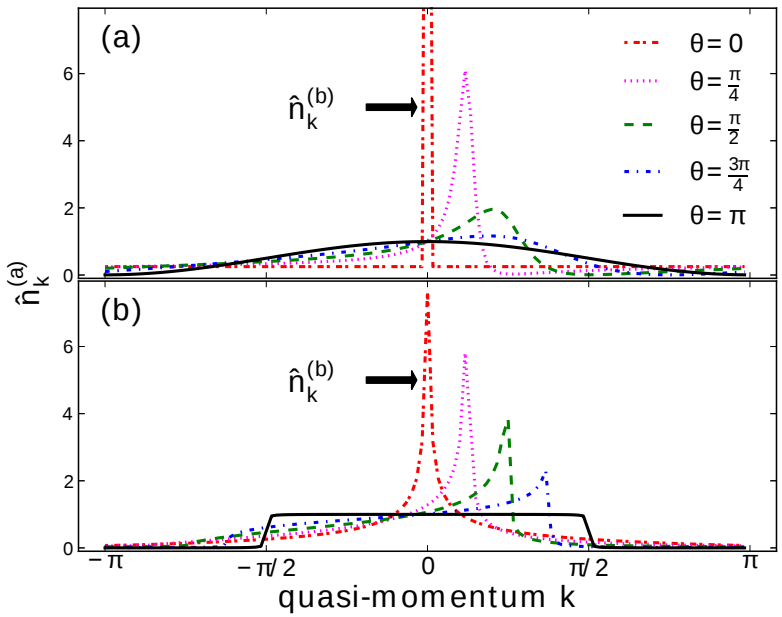
$$\begin{aligned}\langle \hat{n}_k^{(\mathrm{b})} \rangle &= \frac{1}{L} \sum_{ij} e^{\mathrm{i}k(x_i - x_j)} \langle \hat{b}_i^\dagger \hat{b}_j \rangle \,, \\ \langle \hat{b}_i^\dagger \hat{b}_j \rangle &= \delta_{ij} \langle \hat{n}_i \rangle + (1 - \delta_{ij}) \langle \hat{b}_i^\dagger \rangle \langle \hat{b}_j \rangle\end{aligned}$$

- **Anyonic version:**

$$\begin{aligned}\langle \hat{n}_k^{(\mathrm{a})} \rangle &= \frac{1}{L} \sum_{ij} e^{\mathrm{i}k(x_i - x_j)} \langle \hat{a}_i^\dagger \hat{a}_j \rangle \,, \\ \langle \hat{a}_i^\dagger \hat{a}_j \rangle &\xrightarrow{i<j} \langle \hat{b}_i^\dagger e^{\mathrm{i}\theta \hat{n}_i} \rangle \Big(\prod_{i<l<j} \langle e^{\mathrm{i}\theta \hat{n}_l} \rangle \Big) \langle \hat{b}_j \rangle \,, \\ \langle \hat{a}_i^\dagger \hat{a}_j \rangle &\xrightarrow{i=j} \langle \hat{b}_i^\dagger \hat{b}_i \rangle \,, \\ \langle \hat{a}_i^\dagger \hat{a}_j \rangle &\xrightarrow{i>j} \langle e^{-\mathrm{i}\theta \hat{n}_j} \hat{b}_j \rangle \Big(\prod_{j<l<i} \langle e^{-\mathrm{i}\theta \hat{n}_l} \rangle \Big) \langle \hat{b}_i^\dagger \rangle\end{aligned}$$

- **Goal:** dependence on anyon statistical parameter θ

1.5 Hard-Core Anyons: Quasi-Momentum Distributions



$$n_0 = N/L = 0.5\,, \qquad L = 120$$

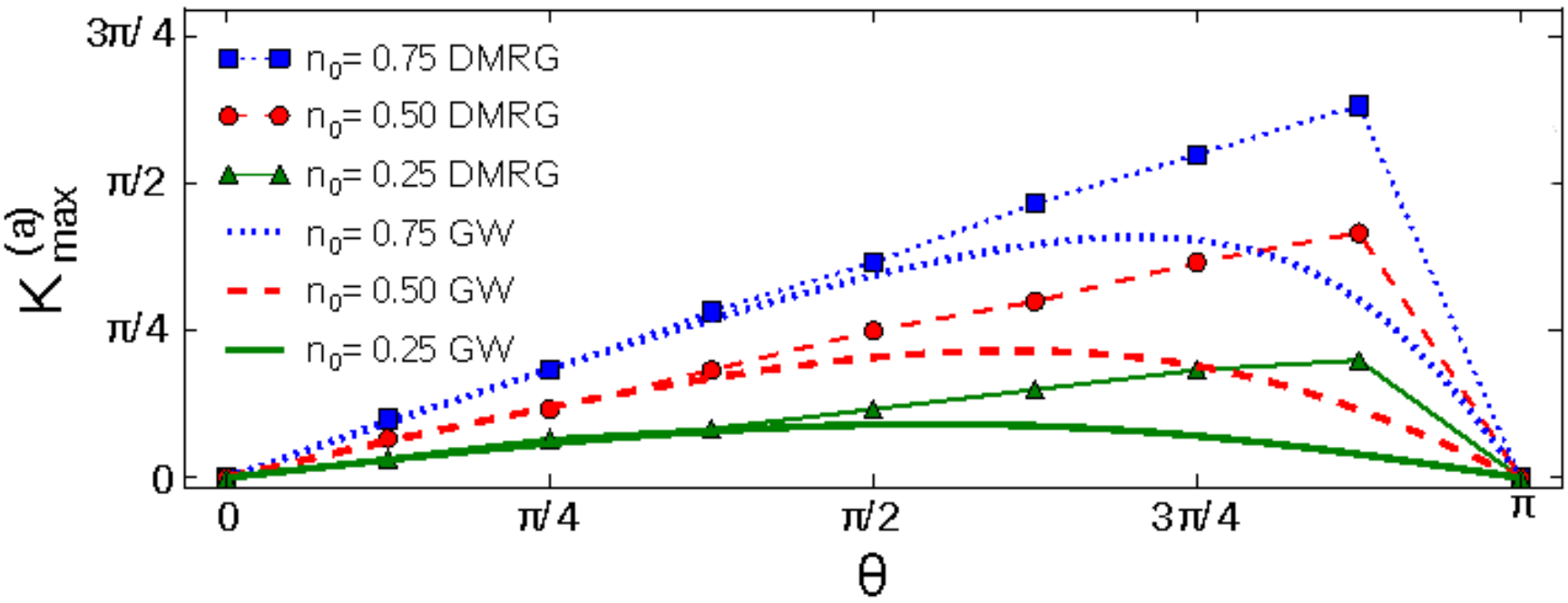
- **(a) Classical GW approach:** $n_{\max} = 1$

$$\langle \hat{n}_k^{(\mathrm{a})} \rangle = n_0 + \frac{n_0(1-n_0)}{L}\left[\frac{(L-1) - L(ze^{-\mathrm{i}k}) + (ze^{-\mathrm{i}k})^L}{e^{\mathrm{i}k}(1-ze^{-\mathrm{i}k})^2} + \mathrm{c.c.}\right]$$

$$z = r\,e^{i\varphi}\,, \qquad r = |1 - n_0 + n_0 e^{\mathrm{i}\theta}|\,, \qquad \varphi = \arg(1 - n_0 + n_0 e^{\mathrm{i}\theta})$$

- **(b) DMRG:** matrix product states, open boundary conditions

1.6 Hard-Core Anyons: Maxima of Distributions



- **Classical GW ($L \to \infty$) smaller shift than DMRG ($L = 120$)**
- **Larger peak shift for bigger n_0**

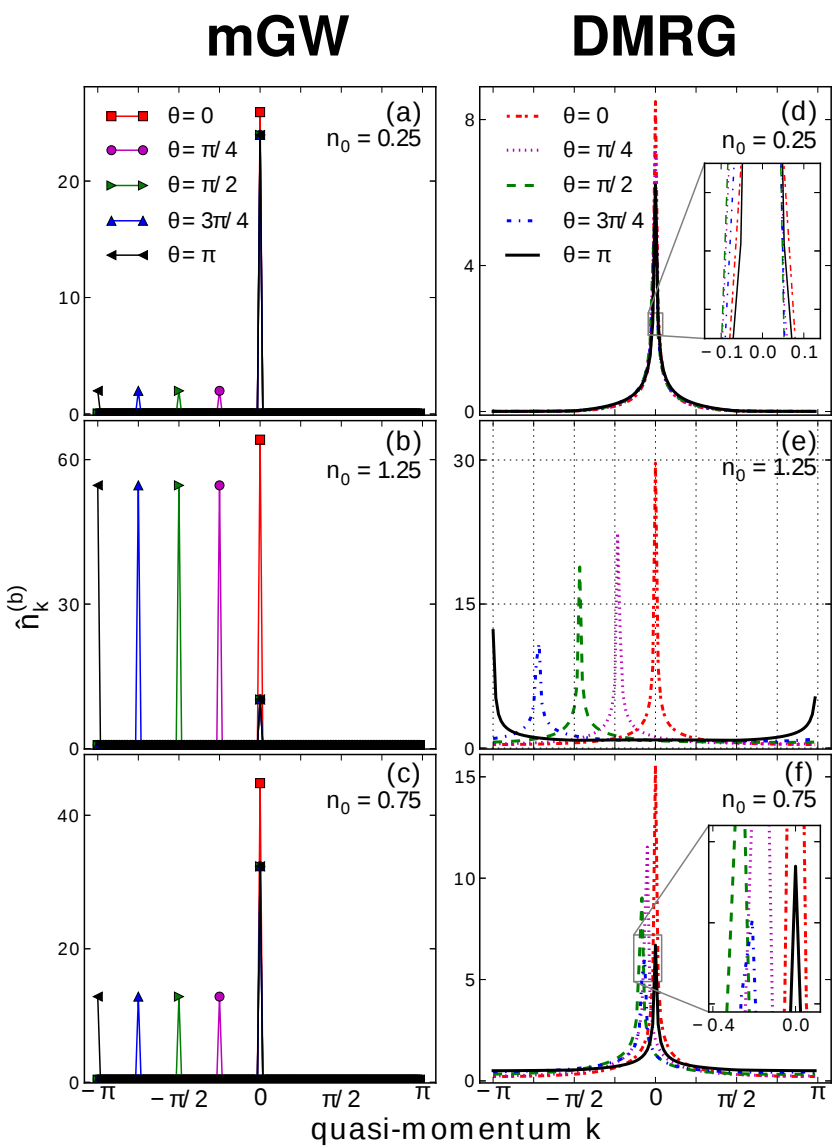
1.7 Soft-Core Anyons: Boson Distributions

- **Modified Gutzwiller:**
  $n_{\max} = 2$

  $$\langle \hat{n}_k^{(\mathrm{b})} \rangle \xrightarrow{L\to\infty} n_0 - C + A\delta(k) + B\delta(k+\theta)$$

- A, B, C **determined via energy extremization**

- **Dependence on** n_0**:**
  - **low density** $n_0 \ll 1$**: peak at** $k = 0$
  - **high density** $n_0 \gg 1$**: peak at** $k = -\theta$



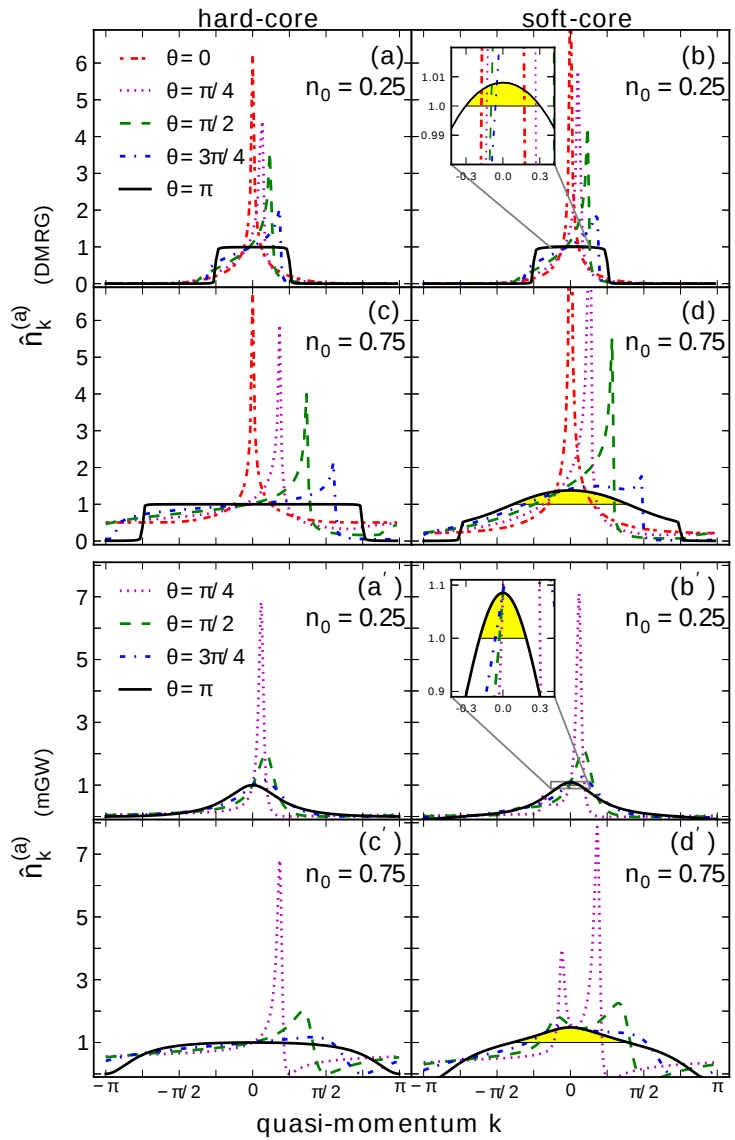
$$J/U = 0.1\,,\, L = 120$$

1.8 Soft-Core Anyons: Anyon Distributions

- **Low density $n_0 \ll 1$:**
  soft-core reproduces hard-core

- **Yellow region $\langle \hat{n}_k^{(a)} \rangle > 1$**
  $\Longrightarrow$ pseudofermions

- **Distribution Maxima shift:**
  depends on Θ and n_0



$$J/U = 0.1\,, L = 120$$

1.9 Outlook

- **Density instead of quasi-momentum distribution:**
  Friedel oscillations (open boundary conditions)
  Sträter, Srivastava, and Eckardt, PRL **117**, 205303 (2016)

- **Quantum phase diagram:**
  Two types of superfluids possible
  Zhang, Fan, Scott, and Zhang, `arXiv:1511.01712`

- **Nearest neighbor interactions:**
  Induce nontrivial topological phases
  Lange, Ejima, and Fehske, `arXiv:1612.00605`

- **Jordan-Wigner Mapping of Anyon-Hubbard to Hubbard model:**
  Independence of exclusion principle and exchange statistics
  Hao, Zhang, and Chen, PRA **79**, 043633 (2009)

- **Finite temperature:**
  Interpolation between Bose-Einstein and Fermi-Dirac statistics

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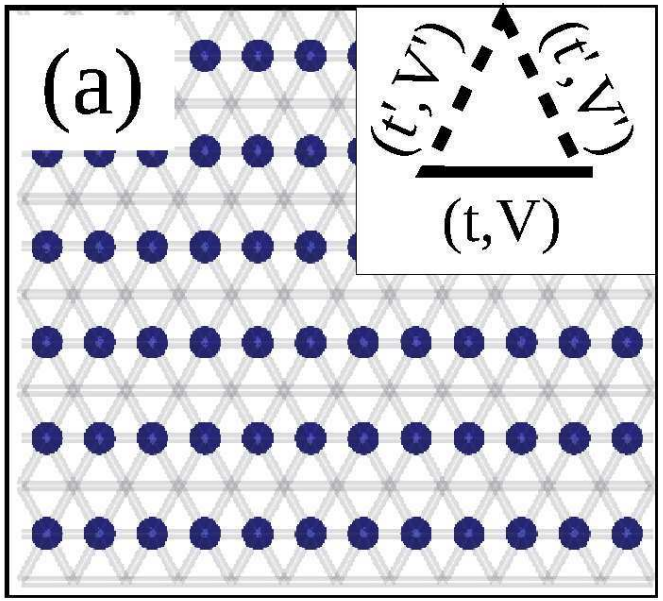
3. Outlook

2.1 Anisotropic Triangular Lattice

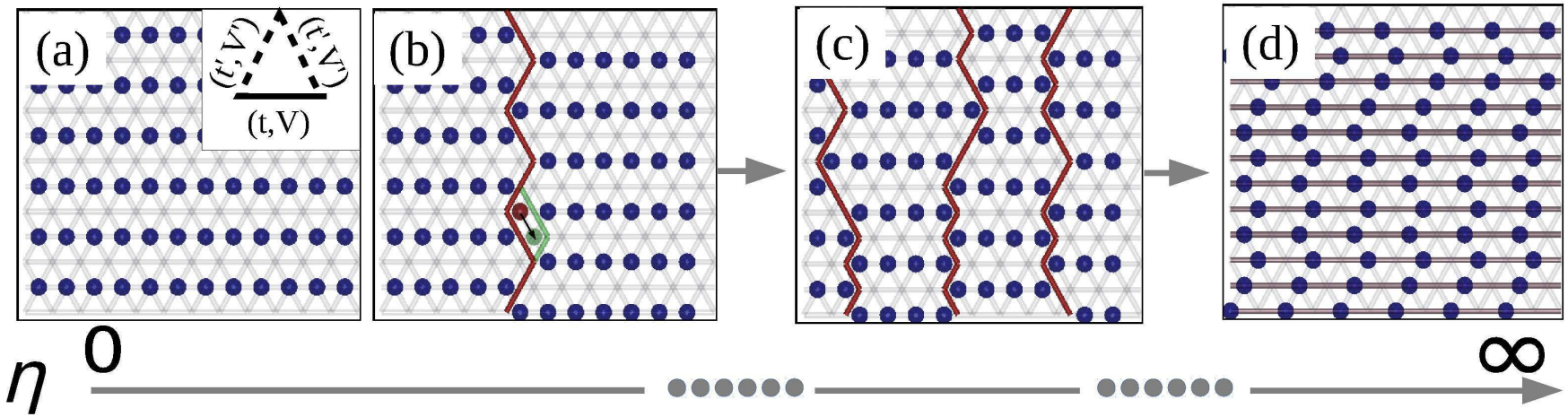
- **Hard-core bosons with anisotropic hopping $t, t' > 0$ and nearest neighbor interactions $V, V' > 0$:**

$$\hat{H} = \sum_{\langle i,j \rangle_x} \left[-t \left(\hat{b}_i^\dagger \hat{b}_j + \text{h.c.} \right) + V \left(\hat{n}_i - 1/2 \right) \left(\hat{n}_j - 1/2 \right) \right]$$
$$+ \sum_{\langle i,j \rangle_\wedge} \left[-t' \left(\hat{b}_i^\dagger \hat{b}_j + \text{h.c.} \right) + V' \left(\hat{n}_i - 1/2 \right) \left(\hat{n}_j - 1/2 \right) \right]$$

- **No chemical potential $\widehat{=}$ half-filling**
- **Number of lattice sites: $L = L_x = L_y$**
- **Anisotropy parameter: $\eta = t/t' = V/V'$**



2.2 Overview



2D solid phase **decoupled 1D chains**

- $\eta = t/t' = V/V' \neq 1$: **Incommensurate supersolid, e.g. incommensurate density order + anisotropic superfluid density**

  Isakov, Chien, Wu, Chen, Chung, Sengupta, and Kim, EPL **87**, 36002 (2009)

- **Results from proliferation of topological defects in form of quantum bosonic domain walls**

  Zhang, Hu, Pelster, and Eggert, PRL **117**, 193210 (2016)

2.3 Topological Defect Model for $\eta < 1$

- **Domain wall energy:**

$$E(N_D) = N_D L_y \left[\frac{V' - V}{2} - \frac{2}{\pi} t' + f\left(\frac{N_D}{L_x}\right) V' \right]$$

Zhang and Eggert, PRL 111, 147201 (2013)

- **Periodic boundary conditions: N_D even**

- **Transition:** $E(N_D = 2M - 2) = E(N_D = 2M)$

- **Appearance of first domain wall pairs ($M = 1$):** $\eta_{c1} = 1 - \frac{4t'}{\pi V'}$

- **Jump points between plateaus:**

$$\eta_M = \eta_{c1} + 2M f\left(\frac{2M}{L_x}\right) - 2(M-1) f\left(\frac{2(M-1)}{L_x}\right) \; ; \quad M = 1, 2, \ldots$$

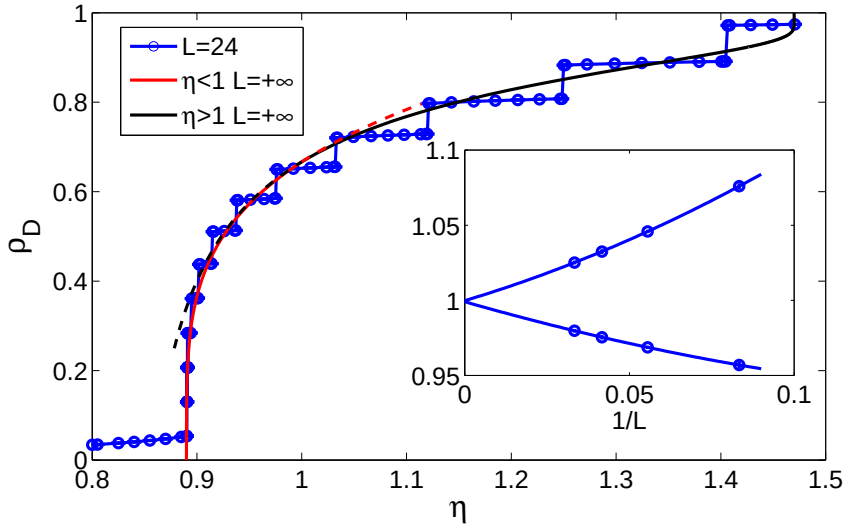
- **Interaction energy:** $f\left(\frac{2M}{L_x}\right) = \sum_{i=1}^{M} \frac{\eta_i}{2M} - \frac{\eta_{c1}}{2}$

2.4 Quantum Monte Carlo

- **Stochastic series expansion**

  Sandvik, PRB **59**, R14157 (1999)

- **Thermalization steps:** 5×10^5
- **Measuring steps:** 10^6
- **Length:** $L_x = L_y = 24$
- **Periodic boundary conditions**



- **Number of domain walls:**
  - **Topological, difficult to change with ordinary QMC updates**
  - **Extension of parallel tempering method**

    Sengupta, Sandvik, and Campbell, PRB **65**, 155113 (2002)

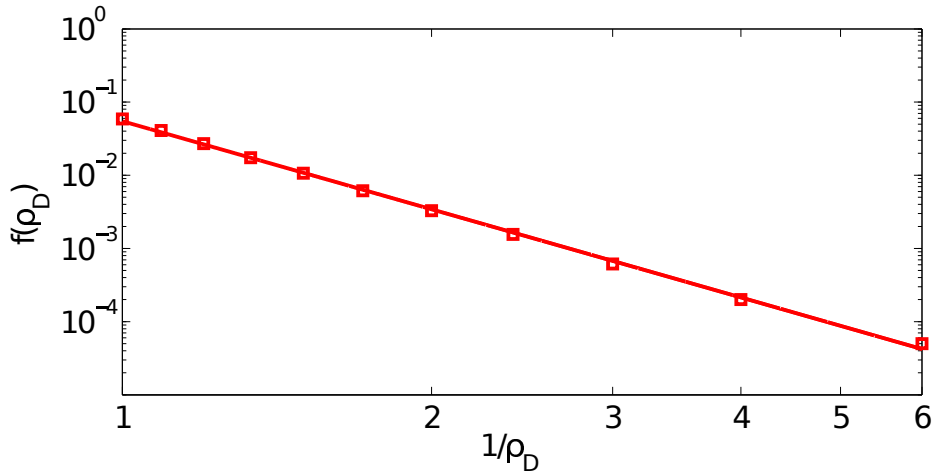
- **Domain wall density:**

$$\rho_D = \sum_{i_y=1}^{L_y} \sum_{i_x=1}^{L_x} \frac{n_{(i_x,i_y)}\bar{n}_{(i_x+1,i_y)} + n_{(i_x+1,i_y)}\bar{n}_{(i_x,i_y)}}{L_x L_y}; \qquad \bar{n} = 1 - n$$

2.5 Topological Defect Model for $\eta < 1$: Continued

- **Jump points from QMC simulations:** $\eta_{c1}, \eta_1, \eta_2, \ldots$
- **Interaction energy:**

$$f\left(\frac{2M}{L_x}\right) = \sum_{i=1}^{M} \frac{\eta_i}{2M} - \frac{\eta_{c1}}{2}$$



- **Fit result:**

$$f(\rho_D) \sim \rho_D^{\alpha}\,, \qquad \alpha = 4 \pm 0.1$$

- **Proportionality coefficient fixed:** $\rho_D(\eta = 1) = \dfrac{2}{3}$
- **Thermodynamic limit $L \Longrightarrow \infty$:**

$$\rho_D(\eta) = \frac{2}{3}\left(\frac{\eta - \eta_{c1}}{1 - \eta_{c1}}\right)^{1/4}, \quad \eta_{c1} \leq \eta < 1$$

- **Appearance of first domain wall pairs:**

$$\eta_{c1} = 1 - \frac{4t'}{\pi V'} \approx 0.89\,, \qquad \frac{t'}{V'} = 0.08$$

2.6 Topological Defect Model for $\eta > 1$

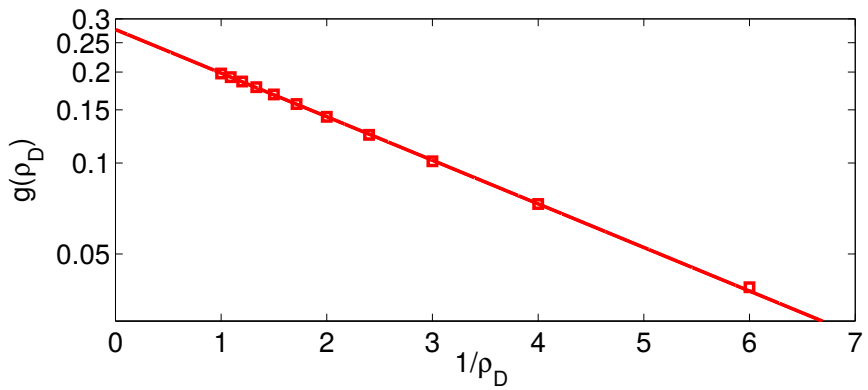
- **Decoupled chain region: density pair excitations**

$$E(N_D) = (L_x - N_D) L_y V' \left[\frac{\eta - 1}{2} - 2\frac{t}{V'} + g\left(\frac{L_x - N_D}{L_x}\right) \right]$$

- **Fit result:**

$$g(\rho_D) \sim \exp\left[-\frac{1}{\gamma(1-\rho_D)} \right]$$

$$\gamma = 3 \pm 0.05$$



- **Thermodynamic limit $L \Longrightarrow \infty$:**

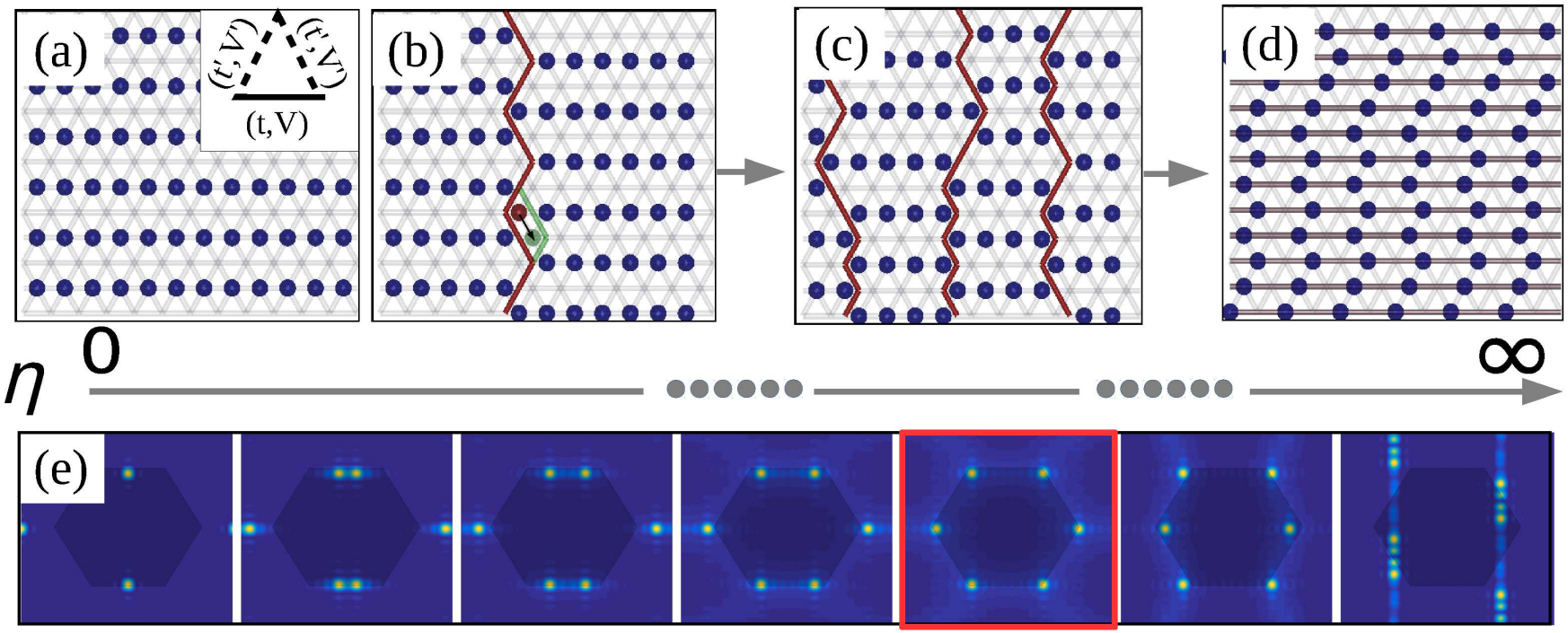
$$\rho_D(\eta) = 1 + \frac{1}{3}\left\{ 1 + W_{-1}\left[-\frac{2(\eta_{c2} - \eta)}{e^2(\eta_{c2} - 1)} \right] \right\}^{-1}, \quad 1 \le \eta < \eta_{c2}$$

W_{-1}: branch -1 of Lambert W function, $z = W(z)e^{W(z)}$

- **Appearance of first density pair excitation:**

$$\eta_{c2} = \frac{1}{1 - 4t'/V'} \approx 1.47\,, \qquad \frac{t'}{V'} = 0.08$$

2.7 Structure Factor



- **Each additional domain wall removes half of density oscillation, thus shifts structure factor peaks by $\pm\pi/L_x$**

- **Microscopic origin of observed incommensurable order:**

$$\mathbf{Q} = \pi(\pm 2 - \rho_D, 0) \quad \text{and} \quad \pi(\pm\rho_D, \pm 2/\sqrt{3})$$

2.8 Anisotropic Superfluid Density

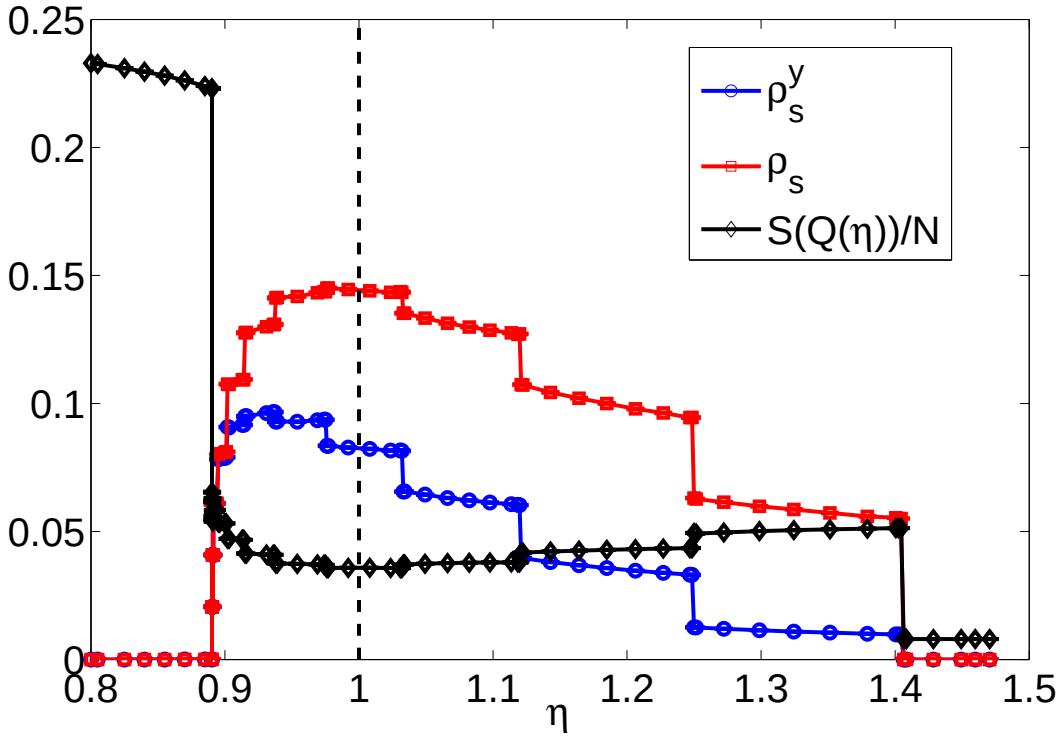
- **Transport of bosons along domain walls**
- **Superfluidity determined via winding numbers**

  Pollock, Ceperley, PRB **36**, 8343 (1987)

$$\rho_s^{x(y)} = W_{x(y)}^2/[4\beta t(t')]$$

$$\rho_s = \rho_s^x + \rho_s^y$$

- $\eta < 1$: ρ_s^y **dominates**
- $\eta = 1$: ρ_s **maximal**
- $\eta > 1$: ρ_s^x **dominates**



- **Superfluid density behaves opposite to structure factor**

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3. Outlook I: Anisotropic Superfluidity

3.1 Superfluid Density as Tensor

- **Linear response theory:** $p_i = VM\left(n_{\mathrm{n}ij}v_{\mathrm{n}j} + n_{\mathrm{s}ij}v_{\mathrm{s}j}\right) + \ldots$

  M. Ueda, *Fundamentals and New Frontiers of Bose-Einstein Condensation* (2010)

- **Dipolar interaction at zero temperature:**
  $\Longrightarrow$ **no anisotropic superfluidity**

  Lima and Pelster, PRA **84**, 041604(R) (2011); PRA **86**, 063609 (2012)

- **Dipolar interaction at finite temperature:**
  $\Longrightarrow$ **Directional dependence of first and second sound velocity**

  Ghabour and Pelster, PRA **90**, 063636 (2014)

- **Dipolar interaction and isotropic disorder at zero temperature:**

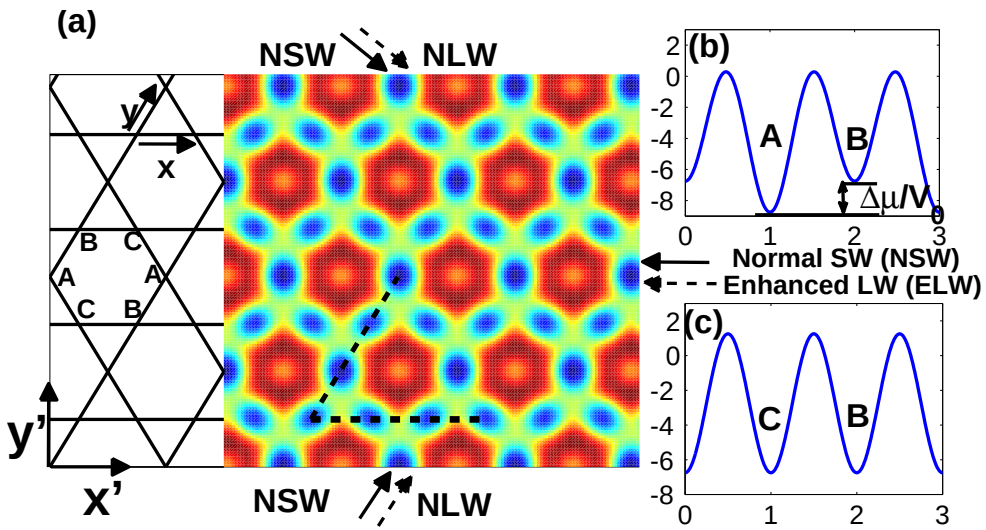
  Krumnow and Pelster, PRA **84**, 021608(R) (2011)

  Nikolić, Balaž, and Pelster, PRA **88**, 013624 (2013)

- **Spin-orbit coupling:**
  $\Longrightarrow$ **Elliptic vortices**

  Devreese, Tempere, and Sá de Melo, PRL **113**, 165304 (2014)

3.2 Proposed Kagome Superlattice

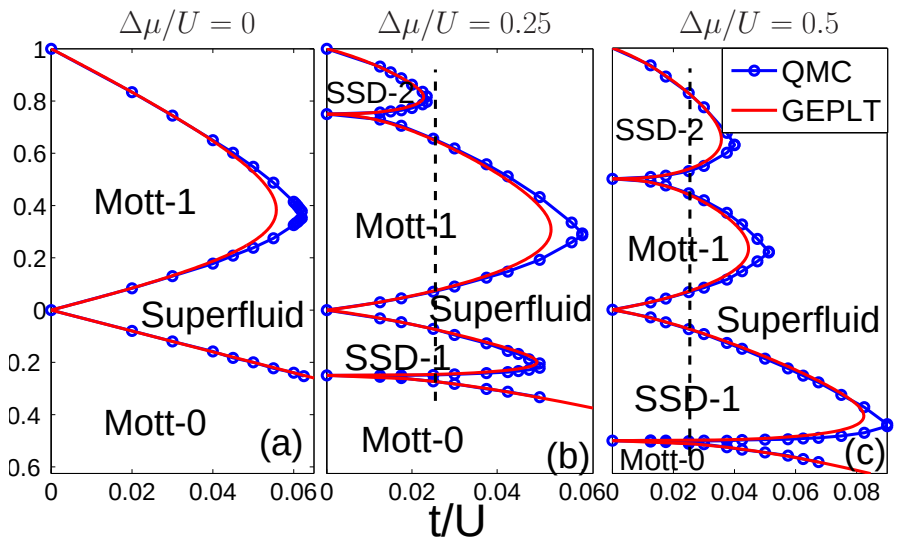


$$\hat{H} = -t \sum_{\langle i,j \rangle} (\hat{a}_i^\dagger \hat{a}_j + \hat{a}_i \hat{a}_j^\dagger) + \frac{U}{2} \sum_i \hat{n}_i (\hat{n}_i - 1) - \mu \sum_i \hat{n}_i - \Delta\mu \sum_{i \in A} \hat{n}_i$$

Zhang, Wang,

Eggert, and Pelster

PRB **92**, 014512 (2015)

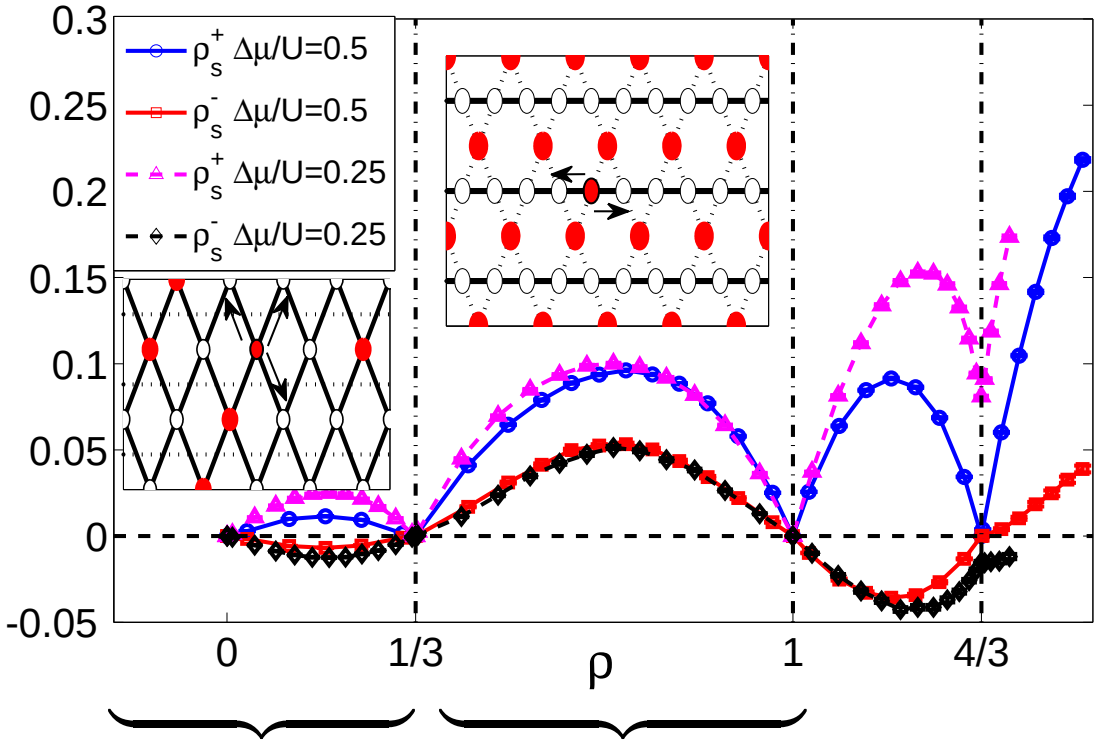


3.3 Tunable Anisotropic Superfluidity

- **Superfluid density via winding number**
  $\rho_s^{x/y} = \langle W_{x/y}^2 / 4\beta t \rangle$
  Pollock and Ceperley, PRB **36**, 8343 (1987)

- **Total superfluid density:**
  $\rho_s^+ = (\rho_s^x + \rho_s^y)/2$

- **Superfluid density difference:**
  $\rho_s^- = \rho_s^x - \rho_s^y$



- $\rho_s^x < \rho_s^y$
- A preferred
- Effective square lattice

- $\rho_s^x > \rho_s^y$
- A full, B/C preferrred
- No supersolid due to artificial symmetry-breaking

Zhang, Wang, Eggert, and Pelster, PRB **92**, 014512 (2015)

3.4 Josephson Sum Rule

- **Superfluid density: tensor** $\Longleftrightarrow$ **condensate density: scalar**

- **Linear response theory, isotropic case:**

$$n_s = \frac{m^2 n_0}{\lim\limits_{\mathbf{k}\to\mathbf{0}} \hbar \mathbf{k}^2 \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} A(\mathbf{k},\omega)}$$

$A(\mathbf{k},\omega)$: spectral function, i.e. Fourier transformed Green's function

B.D. Josephson, PL **21**, 608 (1966)

M. Ueda, *Fundamentals and New Frontiers of Bose-Einstein Condensation* (2010)

- **Consequence for critical exponents:** $\beta_s = \beta_0 - \eta\,\nu$

Hinrichs, Pelster, and Holthaus, APB **113**, 57 (2013)

- **Experimental verfication?**

- **Anisotropic case?**

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Zhang, Hu, Pelster, and Eggert, PRL **117**, 193210 (2016)

3 Outlook II: Exact Quantum Field Mappings

3.5 Mapping Between Quantum Gas Experiments

- **Quantum mechanics: harmonic oscillator $\Longleftrightarrow$ free particle**

Jackiw, Ann. Phys. **129**, 183 (1980)

Cai, Inomata, and Wang, PLA **91**, 331 (1982)

Pelster and Wunderlin, ZPB **89**, 373 (1992)

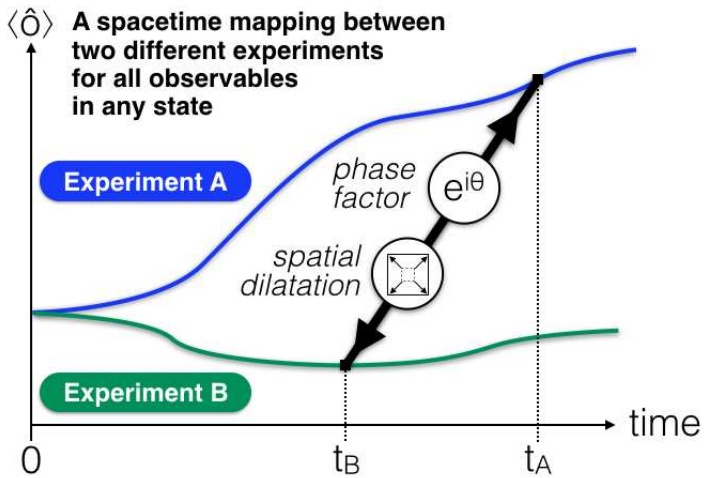
$$\frac{d\tau(t)}{dt} = \lambda^2(t)$$

- **Heisenberg equations of many-body theory:**

$$\hat{\tilde{\psi}}(\mathbf{r},t) = e^{-\frac{iM_n}{2\hbar}\frac{\dot{\lambda}}{\lambda}r^2}\lambda^{D/2}\hat{\psi}(\lambda\mathbf{r},\tau(t))$$

$$\tilde{V}(\mathbf{r},t) = \lambda^2 V(\lambda\mathbf{r},\tau(t)) + \frac{Mr^2}{2}\lambda^3\left(\frac{1}{\lambda^2}\frac{d}{dt}\right)^2\lambda$$

$$\tilde{U}(\mathbf{r},\mathbf{r}',t) = [\lambda(t)]^2 U(\lambda(t)\mathbf{r},\lambda(t)\mathbf{r}',\tau(t))$$



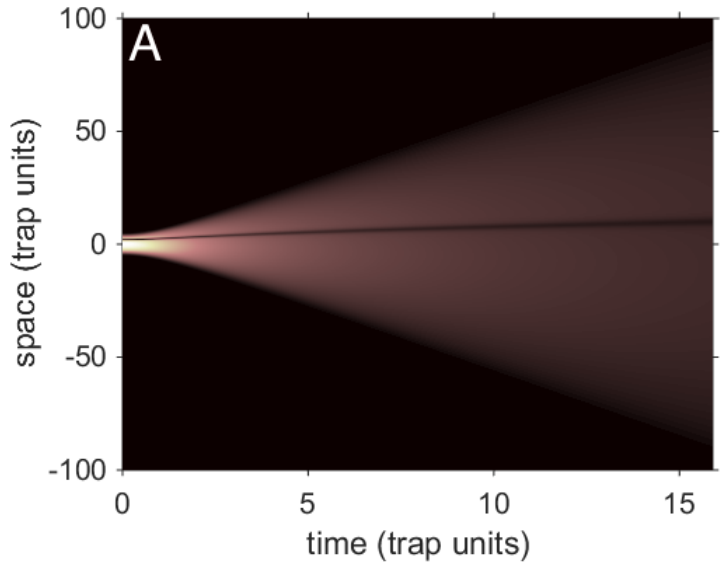
Wamba, Pelster, and Anglin, PRA **94**, 043628 (2016)

3.6 Example: Mappings of 1D GP Equations

- **Experiment A: Time-of-flight**

$$U(x, x', \tau) = g\,\delta(x - x')$$
$$V(x) = 0$$

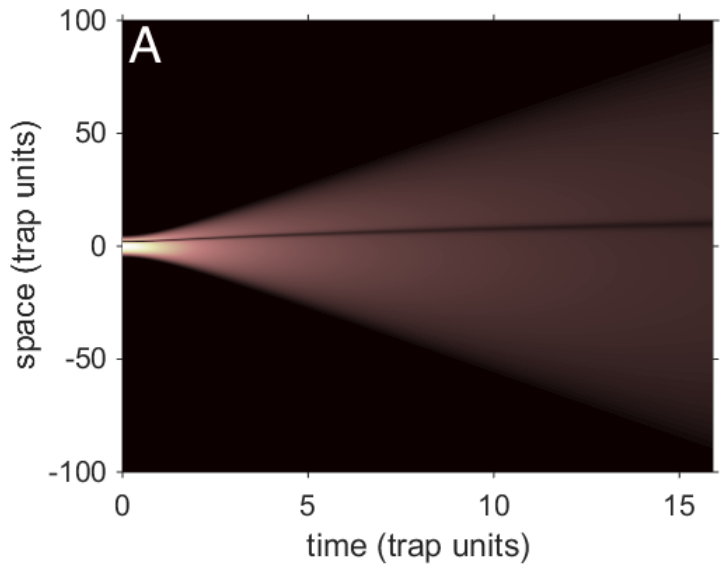
$$\tau(t) = \frac{\tan \omega t}{\omega}$$



- **Experiment B: Ramped interaction**

$$\tilde{U}(x, x', t) = \frac{g}{\cos \omega t}\,\delta(x - x')$$
$$\tilde{V}(x) = \frac{M}{2}\omega^2 x^2$$

- **Common initial state: dark soliton**



Announcements

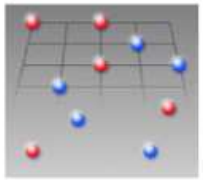
Intensive Week:
Bonn, March 13-17, 2017
Introduction to Topological Insulators and Their Implementations in Artificial Matter Setups

SFB/TR 185

Andrea Alberti, Alexander Altland, Janos Asbóth, Michael Fleischhauer, Fabian Hassler, Netanel Lindner, Reinhard Werner

http://quantum-technologies.iap.uni-bonn.de/topo2017.html

International School:
Tutzing, March 27-30, 2017
Real and Synthetic Magnetism



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Frankfurt – Kaiserslautern - Mainz

Condensed matter systems with variable many-body interactions

Sergej Demokritov, Rembert Duine, Selim Jochim, Carlos Sa de Melo, Stefan Süllow, Christof Wunderlich

http://www-user.rhrk.uni-kl.de/~apelster/Seminar8/index.html