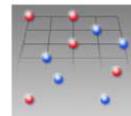
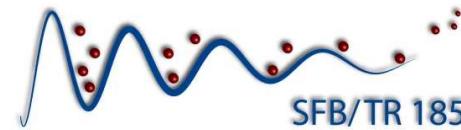


Functional Integral Approach for Trapped Dirty Bosons

Axel Pelster

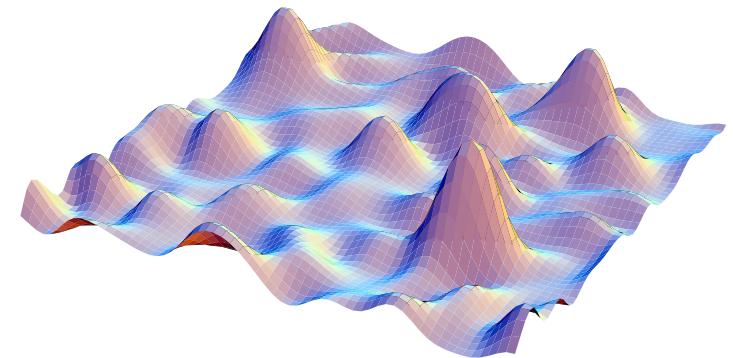


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- 6. Outlook**



1.1 Overview of Set-Ups

- **Superfluid Helium in Porous Media:**

Crooker et al., PRL **51**, 666 (1983)

- **Laser Speckles:**

Lye et al., PRL **95**, 070401 (2005)

Clément et al., PRL **95**, 170409 (2005)

- **Wire Traps:**

Krüger et al., PRA **76**, 063621 (2007)

Fortagh and Zimmermann, RMP **79**, 235 (2007)

- **Localized Atomic Species:**

Gavish and Castin, PRL **95**, 020401 (2005)

Gadway et al., PRL **107**, 145306 (2011)

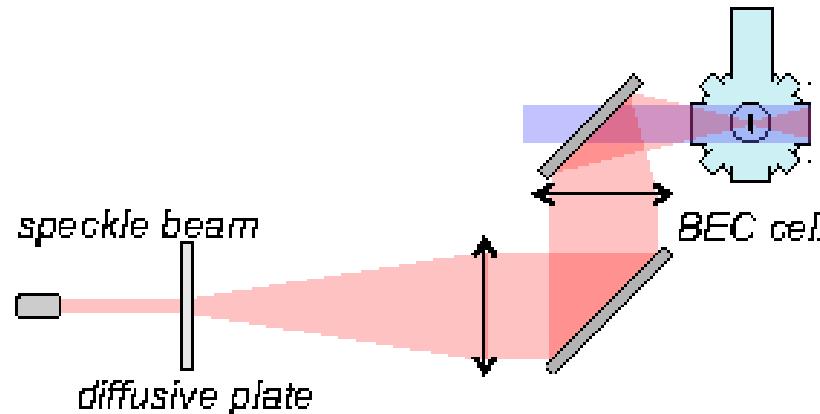
- **Incommensurate Lattices:**

Damski et al., PRL **91**, 080403 (2003)

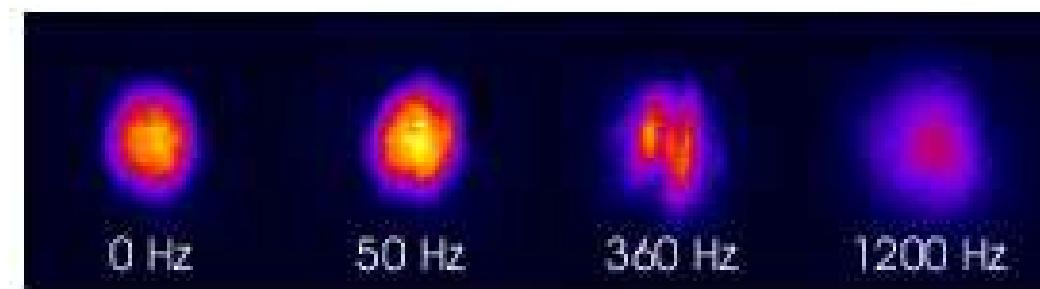
Schulte et al., PRL **95**, 170411 (2005)

1.2 Laser Speckles: Controlled Randomness

Experimental Set-Up:

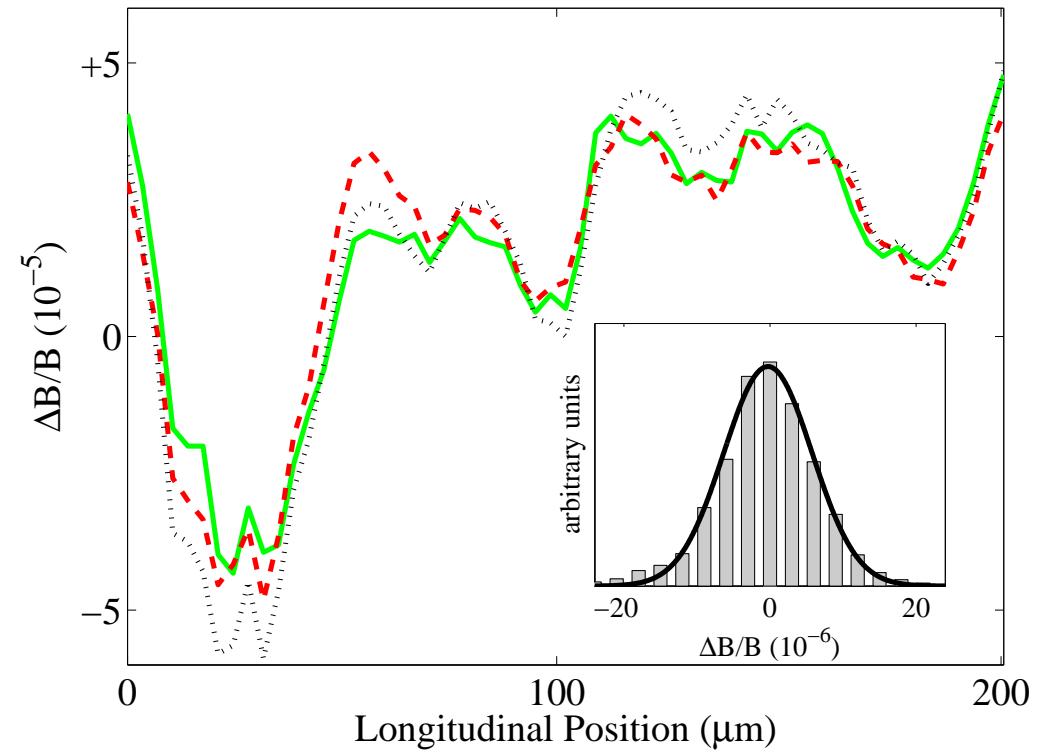
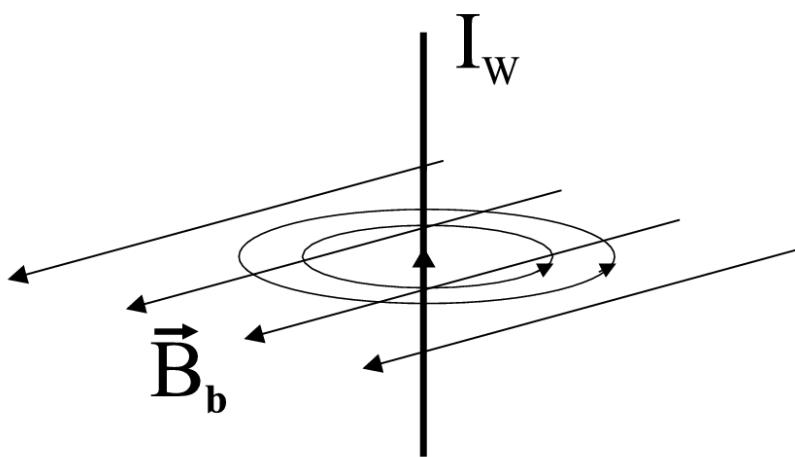


Fragmentation:



Lye et al., PRL 95, 070401 (2005)

1.3 Wire Trap: Undesired Randomness



Distance: $d = 10 \mu\text{m}$

Magnetic Field: 10 G, 20 G, 30 G

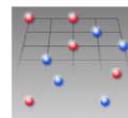
Krüger et al., PRA 76, 063621 (2007)

Wire Width: 100 μm

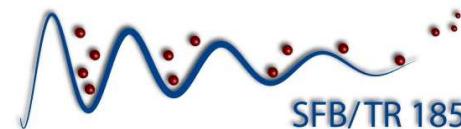
Deviation: $\Delta B/B \approx 10^{-4}$

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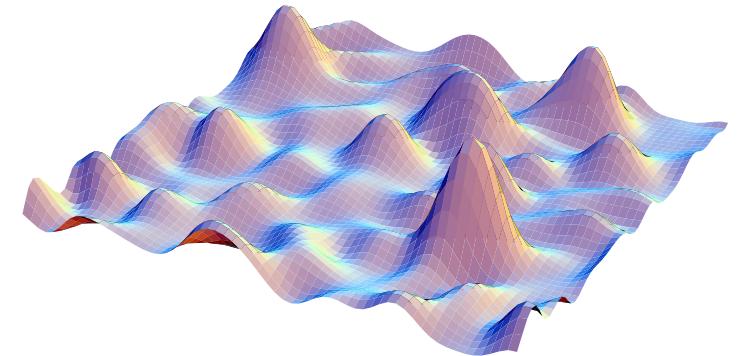


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2.1 Model System

Action of a Bose gas:

$$\begin{aligned}\mathcal{A} = & \int_0^{\hbar\beta} d\tau \int d^3x \left\{ \psi^*(\mathbf{x}, \tau) \left[\hbar \frac{\partial}{\partial \tau} - \frac{\hbar^2}{2M} \Delta + U(\mathbf{x}) + V(\mathbf{x}) - \mu \right] \psi(\mathbf{x}, \tau) \right. \\ & \left. + \frac{g}{2} \psi^*(\mathbf{x}, \tau)^2 \psi(\mathbf{x}, \tau)^2 \right\}\end{aligned}$$

Properties:

- harmonic trap potential: $U(\mathbf{x}) = \frac{M}{2} \sum_{i=1}^3 \omega_i^2 x_i^2$
- disorder potential: $V(\mathbf{x})$; bounded from below, i.e. $V(\mathbf{x}) > V_0$
- chemical potential: μ
- repulsive interaction: $g = \frac{4\pi\hbar^2 a}{M}$
- periodic Bose fields: $\psi(\mathbf{x}, \tau + \hbar\beta) = \psi(\mathbf{x}, \tau)$

2.2 Random Potential

Disorder Ensemble Average:

$$\overline{\bullet} = \int \mathcal{D}V \bullet P[V] , \quad \int \mathcal{D}V P[V] = 1 , \quad P[V < V_0] = 0$$

Assumption:

$$\overline{V(\mathbf{x}_1)} = 0 , \quad \overline{V(\mathbf{x}_1)V(\mathbf{x}_2)} = R^{(2)}(\mathbf{x}_1 - \mathbf{x}_2)$$

Characteristic Functional:

$$\begin{aligned} & \overline{\exp \left\{ i \int d^D x j(\mathbf{x}) V(\mathbf{x}) \right\}} \\ &= \exp \left\{ \sum_{n=2}^{\infty} \frac{i^n}{n!} \int d^D x_1 \cdots \int d^D x_n R^{(n)}(\mathbf{x}_1, \dots, \mathbf{x}_n) j(\mathbf{x}_1) \cdots j(\mathbf{x}_n) \right\} \end{aligned}$$

2.3 Grand-Canonical Potential

Aim:

$$\begin{aligned}\Omega &= -\frac{1}{\beta} \overline{\ln \mathcal{Z}} \\ \mathcal{Z} &= \oint D^2\psi e^{-\mathcal{A}[\psi^*, \psi]/\hbar}\end{aligned}$$

Problem:

$$\overline{\ln \mathcal{Z}} \neq \ln \overline{\mathcal{Z}}$$

Solution: Replica Trick

$$\Omega = -\frac{1}{\beta} \lim_{N \rightarrow 0} \frac{\overline{\mathcal{Z}^N} - 1}{N}$$

Parisi, J. Phys. (France) **51**, 1595 (1990)

Mezard and Parisi, J. Phys. I (France) **1**, 809 (1991)

2.4 Replica Trick

Disorder Averaged Partition Function:

$$\overline{\mathcal{Z}^N} = \overline{\oint \left\{ \prod_{\alpha'=1}^N D^2 \psi_{\alpha'} \right\} e^{-\sum_{\alpha=1}^N \mathcal{A}([\psi_\alpha^*, \psi_\alpha])/\hbar}} = \oint \left\{ \prod_{\alpha=1}^N D^2 \psi_\alpha \right\} e^{-\mathcal{A}^{(N)}/\hbar}$$

Replicated Action:

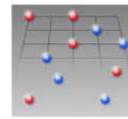
$$\begin{aligned} \mathcal{A}^{(N)} &= \int_0^{\hbar\beta} d\tau \int d^D x \sum_{\alpha=1}^N \left\{ \psi_\alpha^*(\mathbf{x}, \tau) \left[\hbar \frac{\partial}{\partial \tau} - \frac{\hbar^2}{2M} \Delta + U(\mathbf{x}) - \mu \right] \psi_\alpha(\mathbf{x}, \tau) \right. \\ &\quad \left. + \frac{g}{2} |\psi_\alpha(\mathbf{x}, \tau)|^4 \right\} + \sum_{n=2}^{\infty} \frac{1}{n!} \left(\frac{-1}{\hbar} \right)^{n-1} \int_0^{\hbar\beta} d\tau_1 \cdots \int_0^{\hbar\beta} d\tau_n \int d^D x_1 \cdots \int d^D x_n \\ &\quad \times \sum_{\alpha_1=1}^N \cdots \sum_{\alpha_n=1}^N R^{(n)}(\mathbf{x}_1, \dots, \mathbf{x}_n) |\psi_{\alpha_1}(\mathbf{x}_1, \tau_1)|^2 \cdots |\psi_{\alpha_n}(\mathbf{x}_n, \tau_n)|^2 \end{aligned}$$

⇒ **Disorder amounts to attractive interaction for $n = 2$**

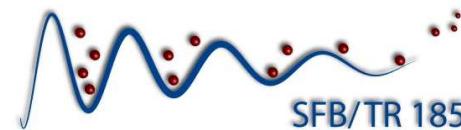
⇒ **Higher-order cumulants negligible in replica limit $N \rightarrow 0$**

Functional Integral Approach for Trapped Dirty Bosons

Axel Pelster

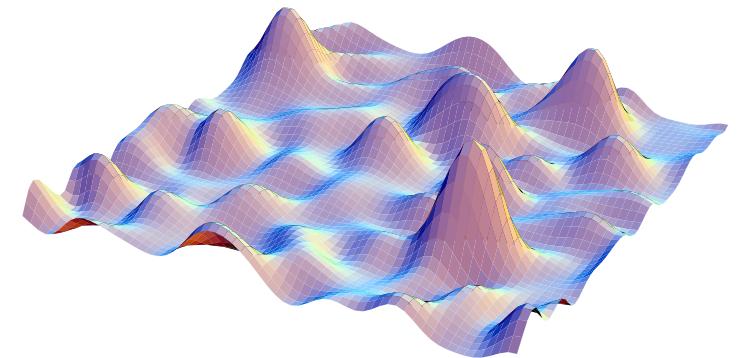


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3.1 Bogoliubov Theory of Dirty Bosons

Assumptions:

homogeneous Bose gas: $U(\mathbf{x}) = 0$

δ -correlated disorder: $\overline{V(\mathbf{x})} = 0, \quad \overline{V(\mathbf{x})V(\mathbf{x}')} = R \delta(\mathbf{x} - \mathbf{x}')$

Condensate Depletion:

$$n_0 = n - \frac{8}{3\sqrt{\pi}} \sqrt{a n_0^3} - \frac{M^2 R}{8\pi^{3/2} \hbar^4} \sqrt{\frac{n_0}{a}}$$

Superfluid Depletion:

$$n_s = n - n_n = n - \frac{4}{3} \frac{M^2 R}{8\pi^{3/2} \hbar^4} \sqrt{\frac{n_0}{a}}$$

Huang and Meng, PRL **69**, 644 (1992)

Falco, Pelster, and Graham, PRA **75**, 063619 (2007)

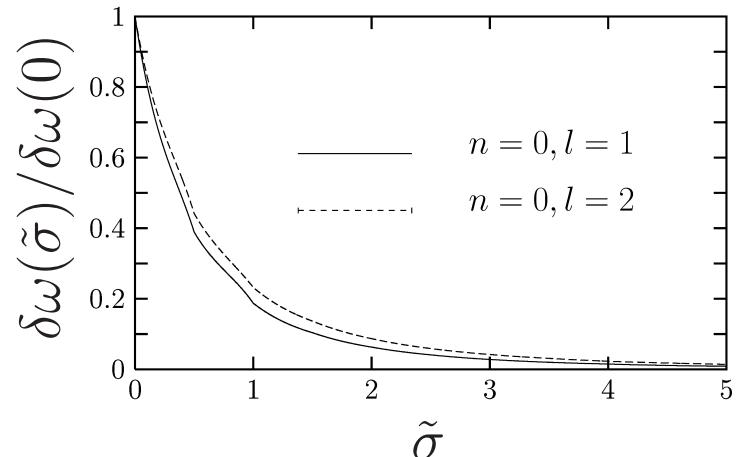
Graham and Pelster, Proc. 9th Int. Conf. Path Integrals, World Scientific (2008)

3.2 Collective Excitations

Typical Values:

Lye et al., PRL **95**, 070401 (2005)

$$\left. \begin{array}{l} \sigma = 10 \text{ } \mu\text{m} \\ R_{\text{TF}} = 100 \text{ } \mu\text{m} \\ l_{\text{HO}} = 10 \text{ } \mu\text{m} \end{array} \right\} \tilde{\sigma} = \frac{\sigma R_{\text{TF}}}{l_{\text{HO}}^2 \sqrt{2}} \approx 7$$



⇒ **Disorder effect vanishes in laser speckle experiment**

Improvement:

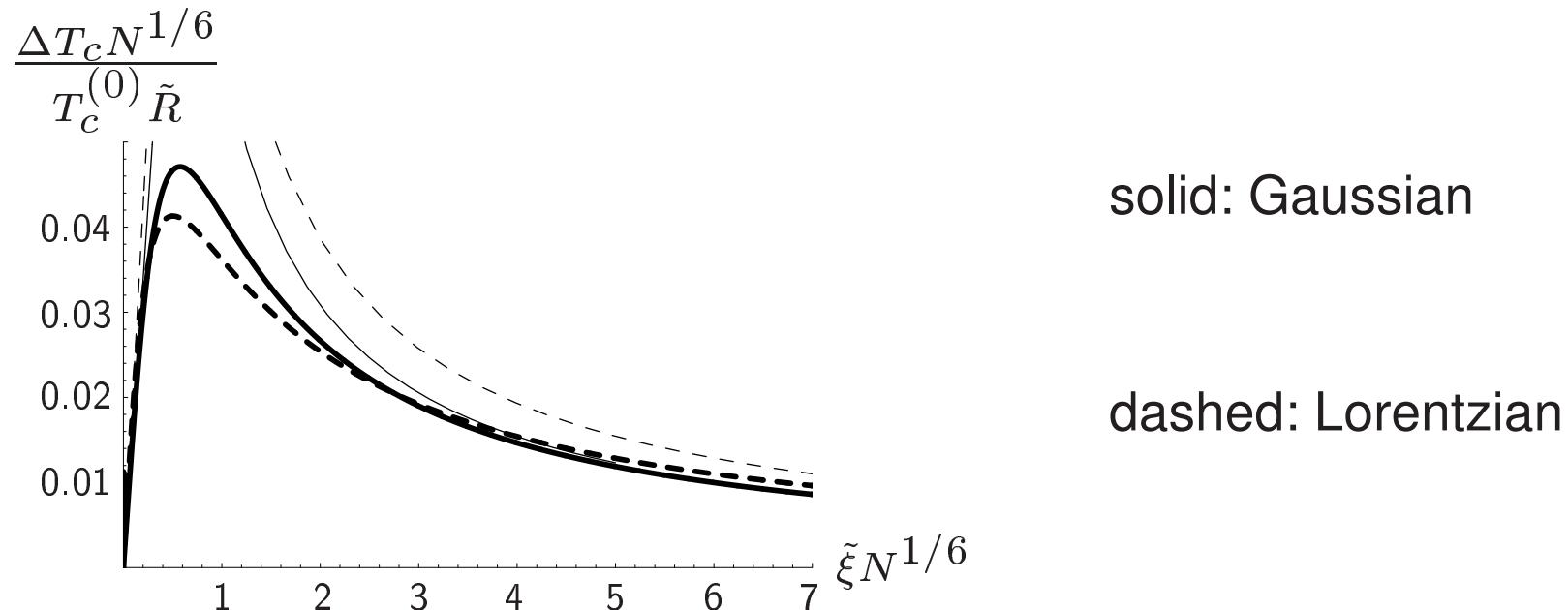
laser speckle setup with correlation length $\sigma = 1 \text{ } \mu\text{m}$

Clément et al., NJP **8**, 165 (2006)

⇒ **Disorder effect should be measurable**

Falco, Pelster, and Graham, PRA **76**, 013624 (2007)

3.3 Shift of Critical Temperature



Length Scale:

$$l_{\text{HO}} = \sqrt{\frac{\hbar}{M\omega_g}} , \quad \omega_g = (\omega_1\omega_2\omega_3)^{1/3}$$

Dimensionless Units:

$$\tilde{\xi} = \frac{\xi}{l_{\text{HO}}} , \quad \tilde{R} = \frac{R}{\left(\frac{\hbar^2}{Ml_{\text{HO}}^2}\right)^2 l_{\text{HO}}^3} = \frac{M^{3/2}R}{\hbar^{7/2}\omega_g^{1/2}}$$

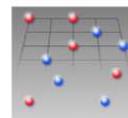
Timmer, Pelster, and Graham, EPL **76**, 760 (2006)

3.4 Superfluid Density as Tensor

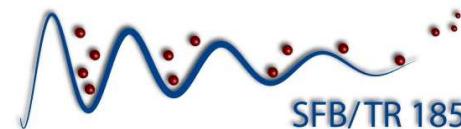
- **Linear response theory:** $p_i = VM(n_{nij}v_{nj} + n_{sij}v_{sj}) + \dots$
M. Ueda, *Fundamentals and New Frontiers of Bose-Einstein Condensation* (2010)
- **Dipolar interaction at zero temperature:**
⇒ **no anisotropic superfluidity**
Lima and Pelster, PRA **84**, 041604(R) (2011); PRA **86**, 063609 (2012)
- **Dipolar interaction at finite temperature:**
⇒ **Directional dependence of first and second sound velocity**
Ghabour and Pelster, PRA **90**, 063636 (2014)
- **Dipolar interaction and isotropic disorder at zero temperature:**
Krumnow and Pelster, PRA **84**, 021608(R) (2011)
Nikolić, Balaž, and Pelster, PRA **88**, 013624 (2013)
- **Condensate depletion larger than parallel superfluid depletion:**
⇒ **Finite localization time**
Graham and Pelster, IJBC **19**, 2745 (2009)

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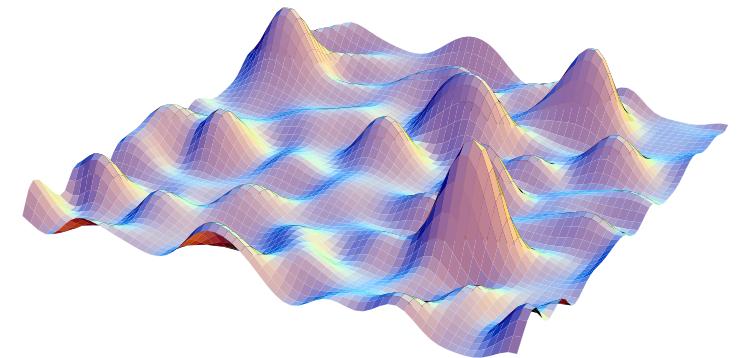


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4.1 Order Parameters

Definition:

$$\lim_{|\mathbf{x}-\mathbf{x}'| \rightarrow \infty} \overline{\langle \psi(\mathbf{x}, \tau) \psi^*(\mathbf{x}', \tau) \rangle} = n_0$$
$$\lim_{|\mathbf{x}-\mathbf{x}'| \rightarrow \infty} \overline{|\langle \psi(\mathbf{x}, \tau) \psi^*(\mathbf{x}', \tau) \rangle|^2} = (n_0 + q)^2$$

Note:

q is similar to Edwards-Anderson order parameter of spin-glass theory

Hartree-Fock Mean-Field Theory:

Self-consistent determination of n_0 and q for $R(\mathbf{x} - \mathbf{x}') = R \delta(\mathbf{x} - \mathbf{x}')$

Phase Classification:

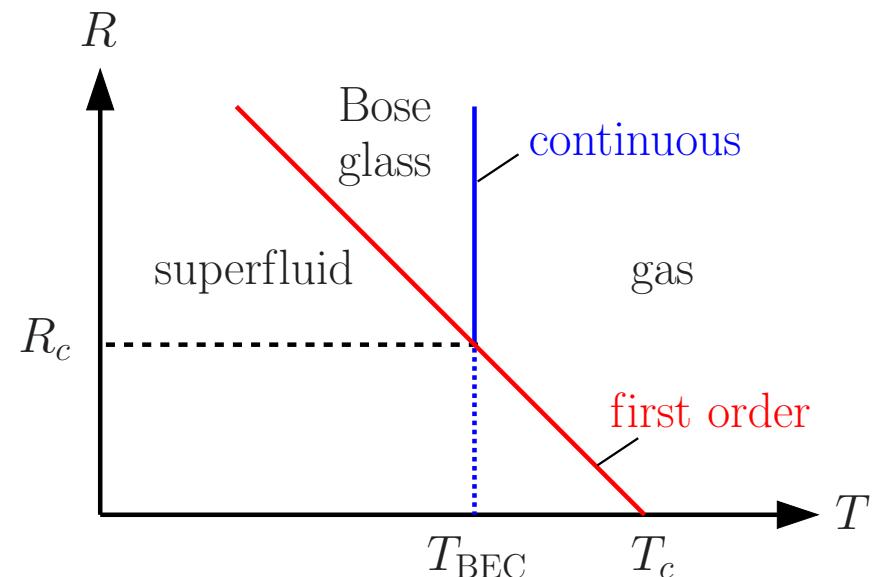
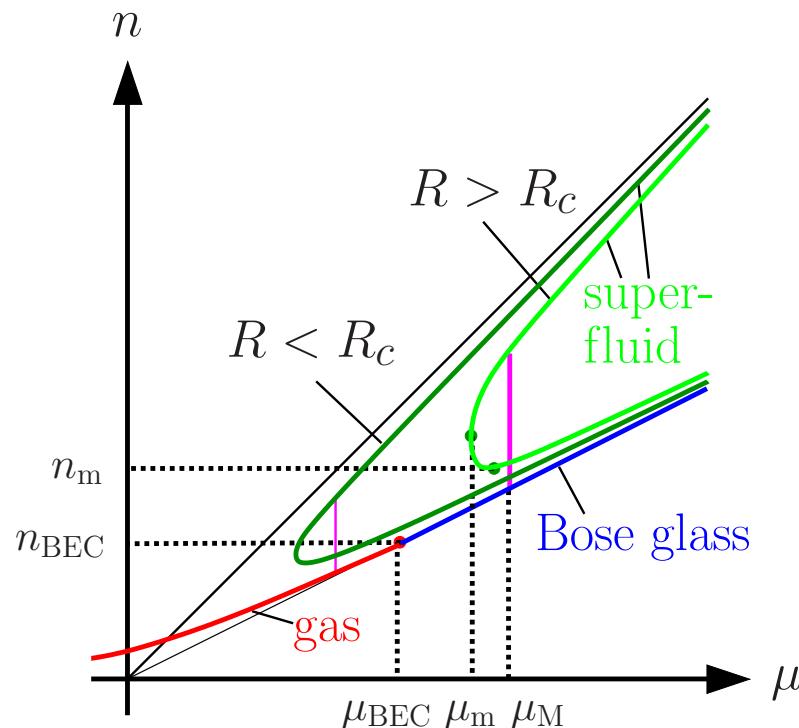
gas	Bose glass	superfluid
$q = n_0 = 0$	$q > 0, n_0 = 0$	$q > 0, n_0 > 0$

4.2 Hartree-Fock Results

Isotherm: $T = \text{const.}$

disorder strength $R = \text{const.}$

Phase Diagram: $\mu = \text{const.}$



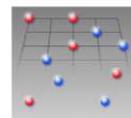
Graham and Pelster, IJBC **19**, 2745 (2009)

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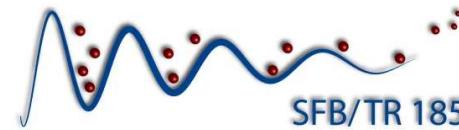
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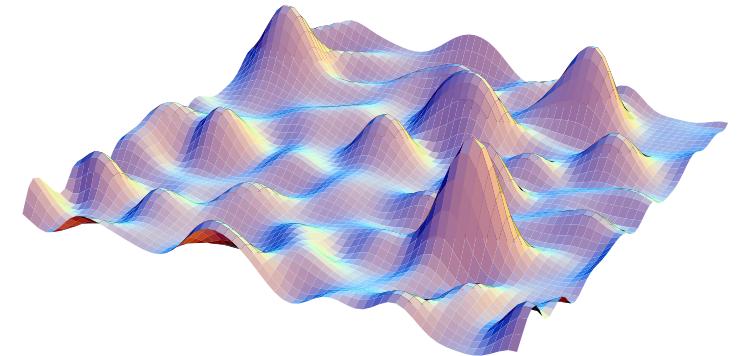


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5.1 Self-Consistency Equations

$$n(\mathbf{x}) = n_0(\mathbf{x}) + q(\mathbf{x}) + n_{\text{th}}(\mathbf{x})$$

$$\left[-\mu + 2gn(\mathbf{x}) + V(\mathbf{x}) - D Q_0(\mathbf{x}) - gn_0(\mathbf{x}) - \frac{\hbar^2}{2m} \Delta \right] \sqrt{n_0(\mathbf{x})} = 0$$

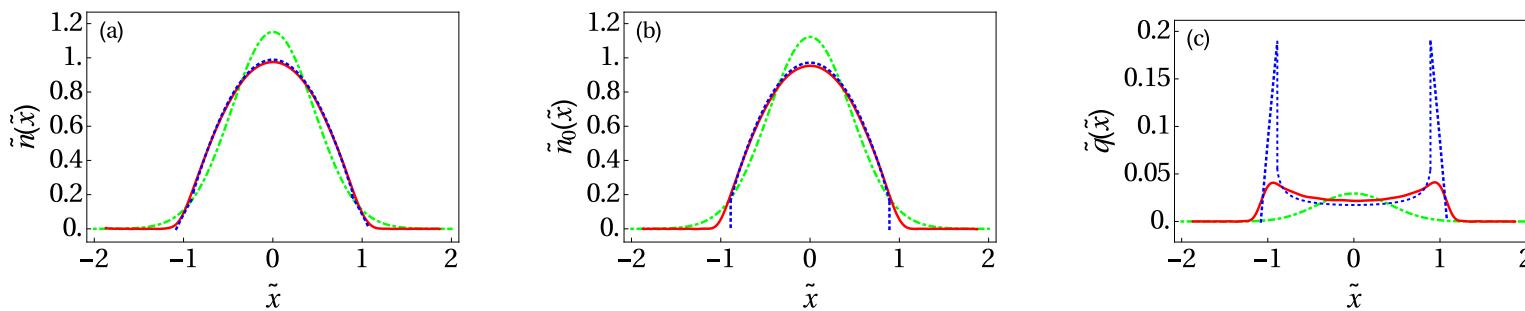
$$q(\mathbf{x}) = D\Gamma \left(2 - \frac{n}{2} \right) \left(\frac{M}{2\pi\hbar^2} \right)^{n/2} \frac{n_0(\mathbf{x}) + q(\mathbf{x})}{[-\mu + 2gn(\mathbf{x}) + V(\mathbf{x}) - D Q_0(\mathbf{x})]^{2-n/2}}$$

$$Q_m(\mathbf{x}) = \Gamma \left(1 - \frac{n}{2} \right) \left(\frac{M}{2\pi\hbar^2} \right)^{n/2} [-i\hbar\omega_m - \mu + 2gn(\mathbf{x}) + V(\mathbf{x}) - D Q_m(\mathbf{x})]^{1-n/2}$$

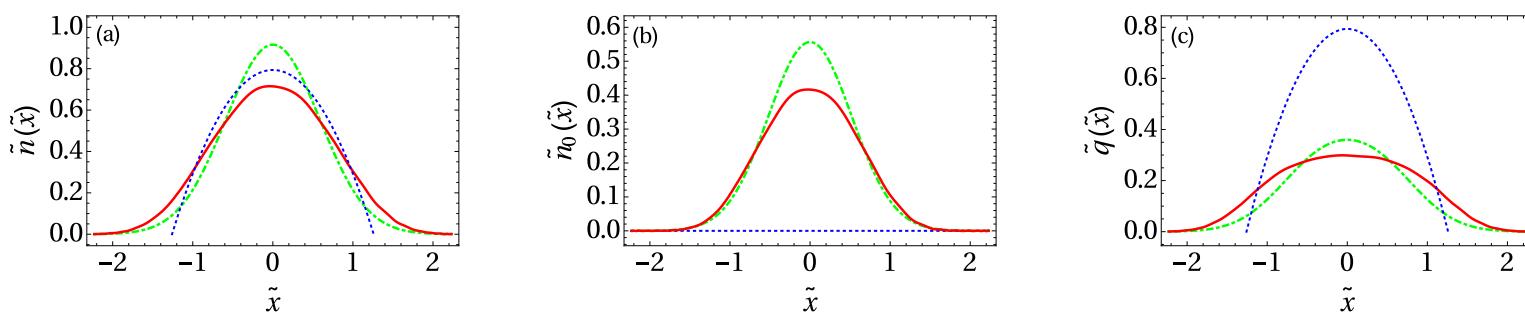
$$n_{\text{th}}(\mathbf{x}) = \lim_{\eta \downarrow 0} \sum_{m=-\infty}^{\infty} e^{i\omega_m \eta} \frac{Q_m(\mathbf{x})}{\beta}, \quad \omega_m = \frac{2\pi m}{\hbar\beta}$$

5.2 Quasi-1d Case, T=0

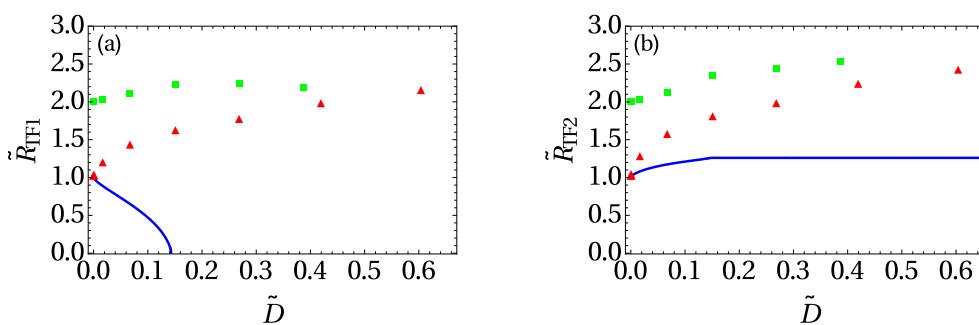
- **Weak disorder:** $\tilde{D} = 0.016$



- **Stronger disorder:** $\tilde{D} = 0.386$



- **Thomas-Fermi radii:**

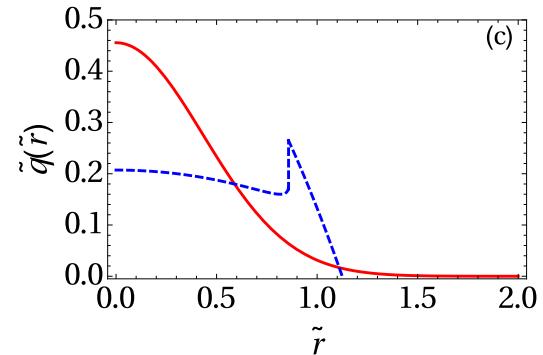
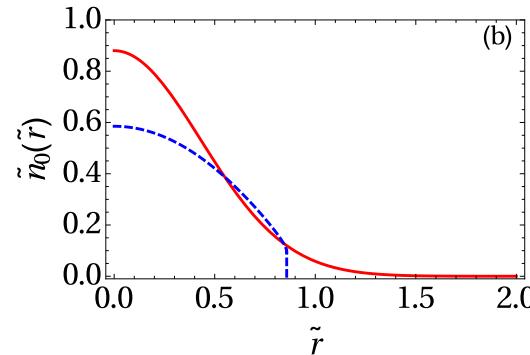
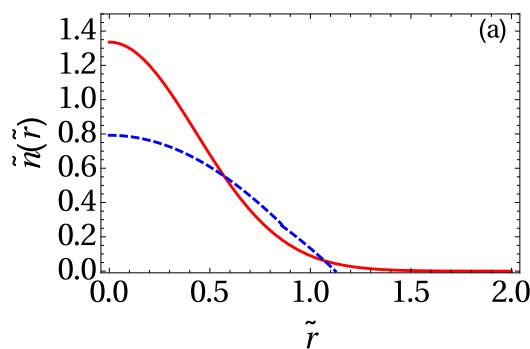


Khellil, Balaz, and Pelster, NJP **18**, 063003 (2016)

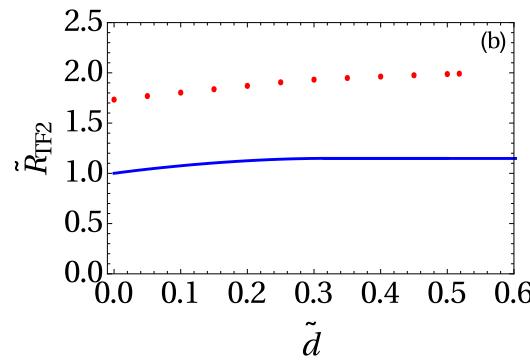
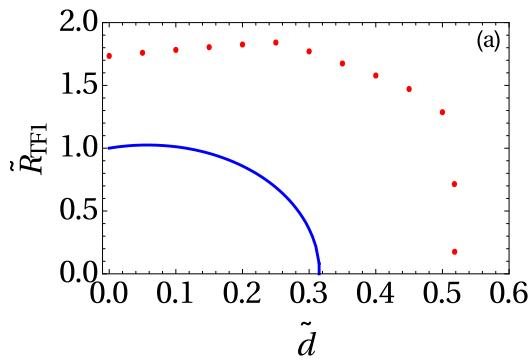
Green \implies Variational
Red \implies Numerical
Blue \implies Analytical

5.3 Isotropic 3d Case, T=0

- **Density profiles:** $\tilde{d} = 0.2$



- **Thomas-Fermi radii:**



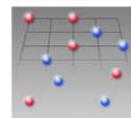
Red \implies Variational
Blue \implies Analytical

Khellil and Pelster, arXiv:1512.04870

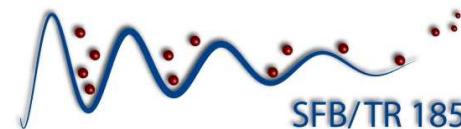
Nattermann and Pokrovsky, PRL 100, 060402 (2008): $\tilde{d}_c = 0.115$

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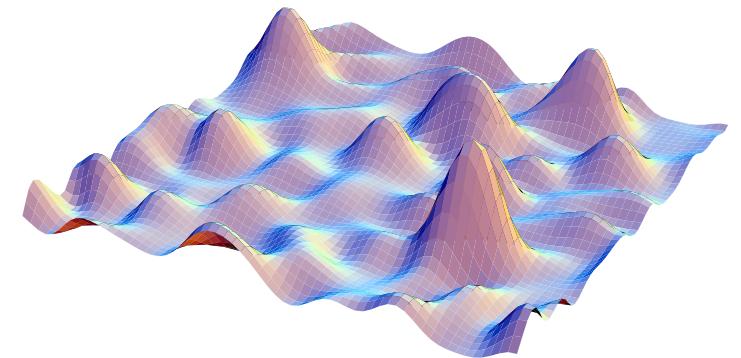


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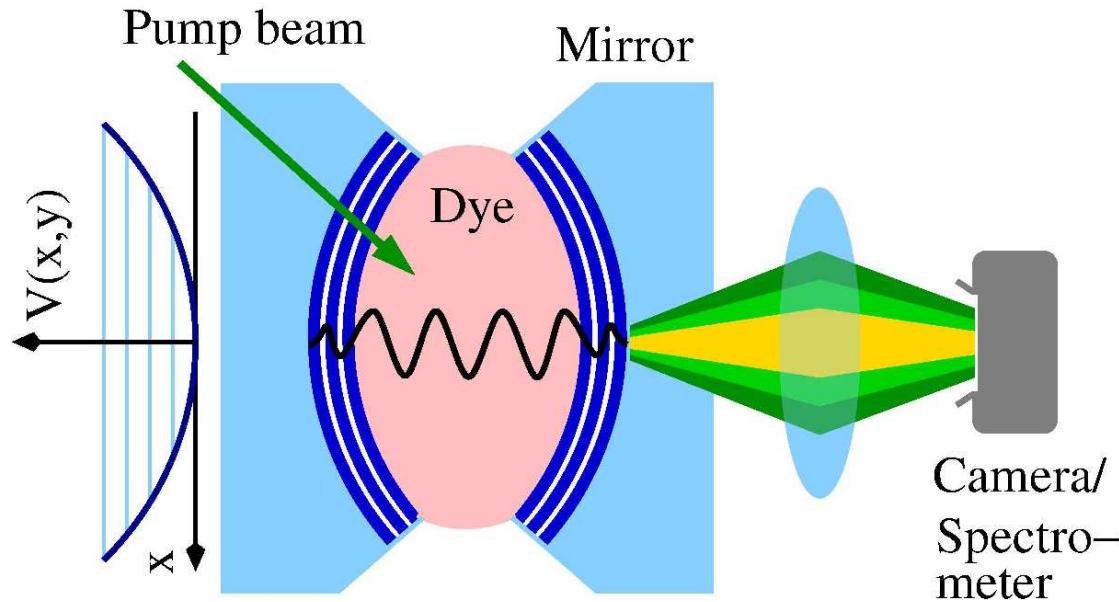
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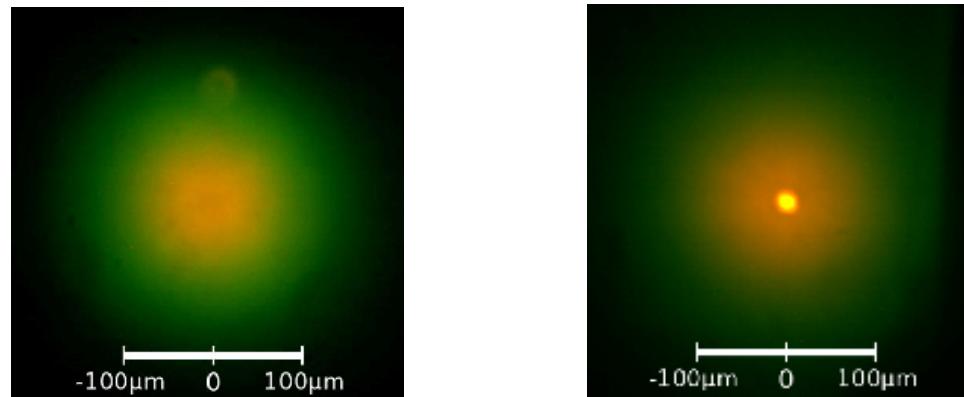


6.1 Bose-Einstein Condensation of Light

Set-Up



Result



Klaers, Schmitt, Vewinger, and Weitz, Nature **468**, 545 (2010)

Pelster, Physik-Journal **10**, Nr. 1, 20 (2011); Physik Journal **13**, Nr. 3, 20 (2014)

Kopylov, Radonjić, Brandes, Balaž, and Pelster, PRA **92**, 063832 (2015)

6.2 Mapping Between Quantum Gas Experiments

- **Time transformation:** $\frac{d\tau(t)}{dt} = \lambda^2(t)$

Jackiw, Ann. Phys. **129**, 183 (1980)

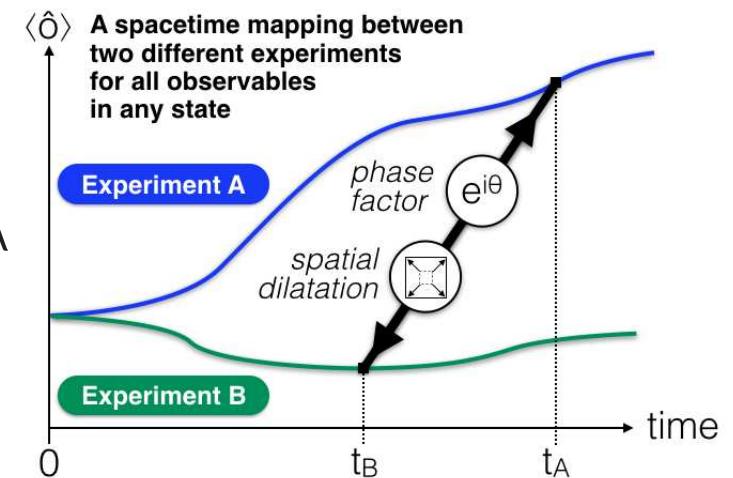
Cai, Inomata, and Wang, PLA **91**, 331 (1982)

- **Transformation formulas:**

$$\hat{\tilde{\psi}}(\mathbf{r}, t) = e^{-\frac{iMn}{2\hbar}\dot{\lambda}r^2} \lambda^{D/2} \hat{\psi}(\lambda\mathbf{r}, \tau(t))$$

$$\tilde{V}(\mathbf{r}, t) = \lambda^2 V(\lambda\mathbf{r}, \tau(t)) + \frac{Mr^2}{2} \lambda^3 \left(\frac{1}{\lambda^2} \frac{d}{dt} \right)^2 \lambda$$

$$\tilde{U}(\mathbf{r}, \mathbf{r}', t) = [\lambda(t)]^2 U(\lambda(t)\mathbf{r}, \lambda(t)\mathbf{r}', \tau(t))$$



Wamba, Pelster, and Anglin, PRA **94**, 043628 (2016)