Controlling the Phase in a Neuronal Feedback Loop Through Asymmetric Temporal Delays

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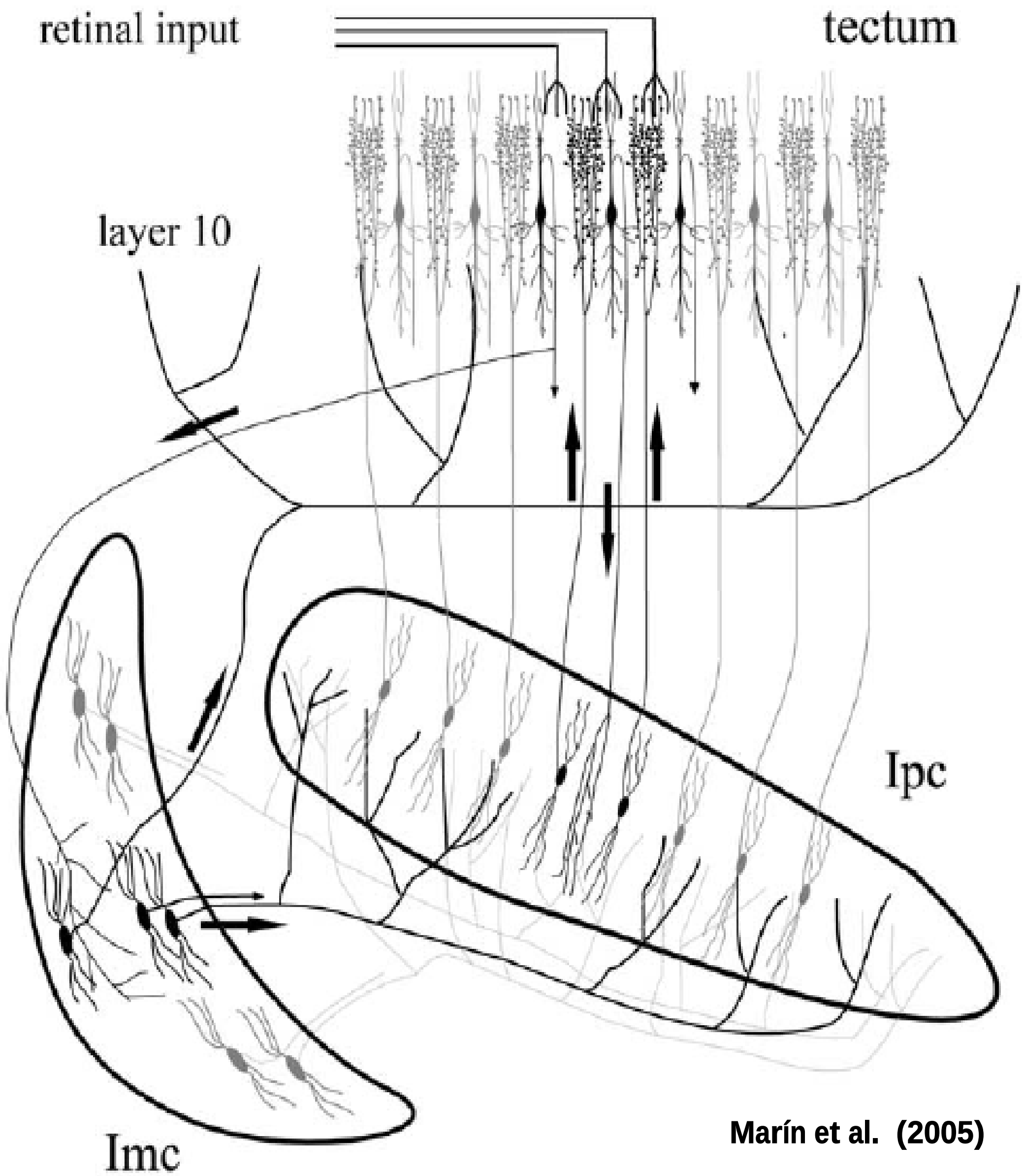
- Two-Neuron System with Asymmetric Delays

  Frequency of Limit Cycle
  Phase Shift
  Variational Perturbation Theory (VPT)

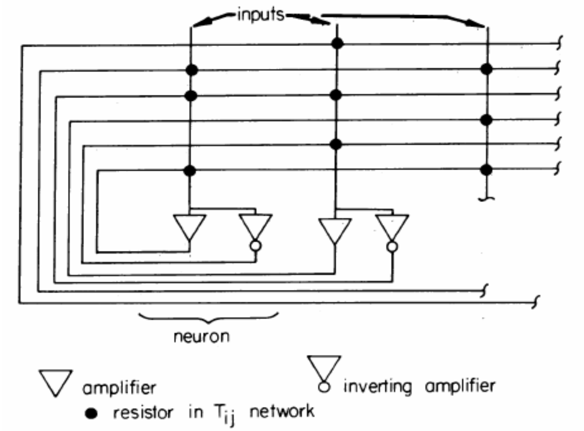
Feedback in the Avian Visual System



Marín et al. (2005)

Time-Continuous Model of Neuron Dynamics by Hopfield

- Leaky neuron with external input and input from other neurons:



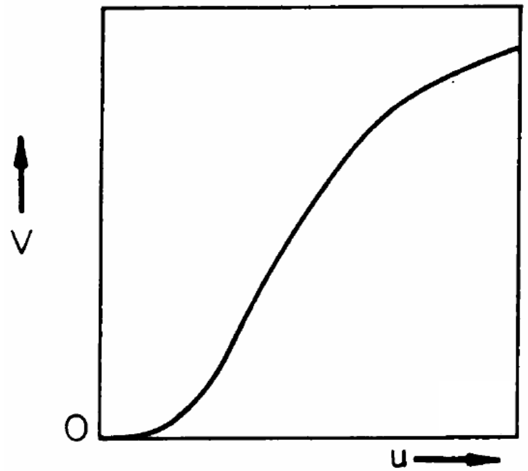
$$C_i \frac{du_i}{dt} = \sum_j T_{ij} V_j - \frac{u_i}{R_i} + I_i$$

Hopfield (1984)

u_i : input voltage, V_j : output voltage, T_{ij}: synaptic interconnection matrix

- Nonlinear Transfer Function:

$$V_i = g(u_i)$$



Two-Neuron Model With Delay

- Model Equation:

$$\begin{aligned}\frac{du_1(t)}{dt} &= -u_1(t) + a_1 \tanh[u_2(t-\tau_2)] \\ \frac{du_2(t)}{dt} &= -u_2(t) + a_2 \tanh[u_1(t-\tau_1)]\end{aligned}$$

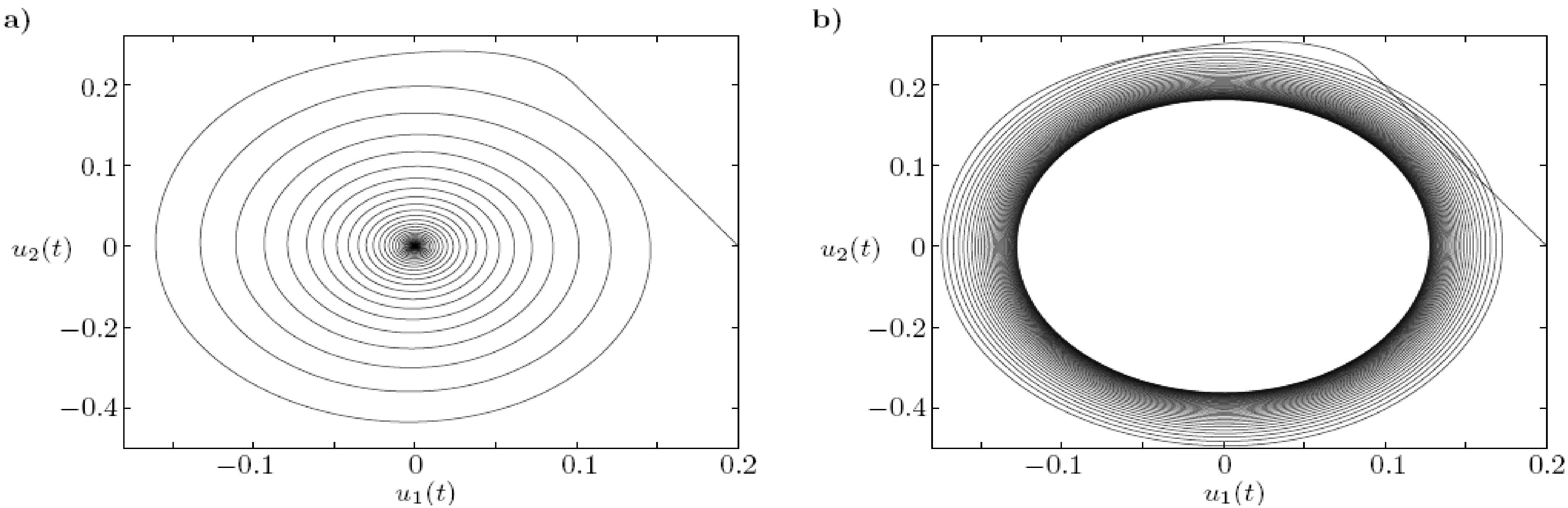
- Characteristic Equation:

$$(\lambda+1)^2 - a_1 a_2 e^{-\lambda(\tau_1+\tau_2)} = 0$$

- Supercritical Hopf-Bifurcation:

$$\tau \equiv \frac{\tau_1+\tau_2}{2} \geq \tau_0 \equiv \frac{1}{2\omega_0} \sin^{-1}\left(-\frac{2\omega_0}{a_1 a_2}\right)$$

$$\omega_0 = \sqrt{|a_1 a_2| - 1} \qquad a_1 = -1,\ a_2 = 2$$



Resumming Divergent Weak-Coupling Series via VPT

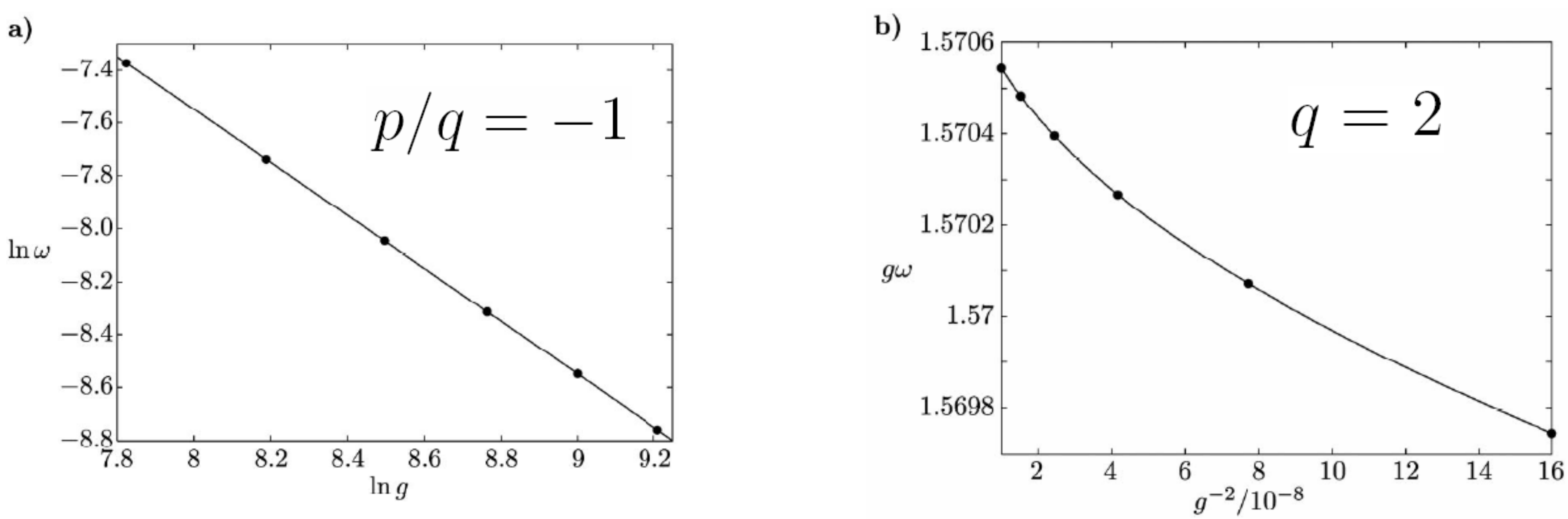
- Expansion Parameter: $\epsilon = \sqrt{\tau - \tau_0}$

- Weak-Coupling Series — Strong-Coupling Expansion

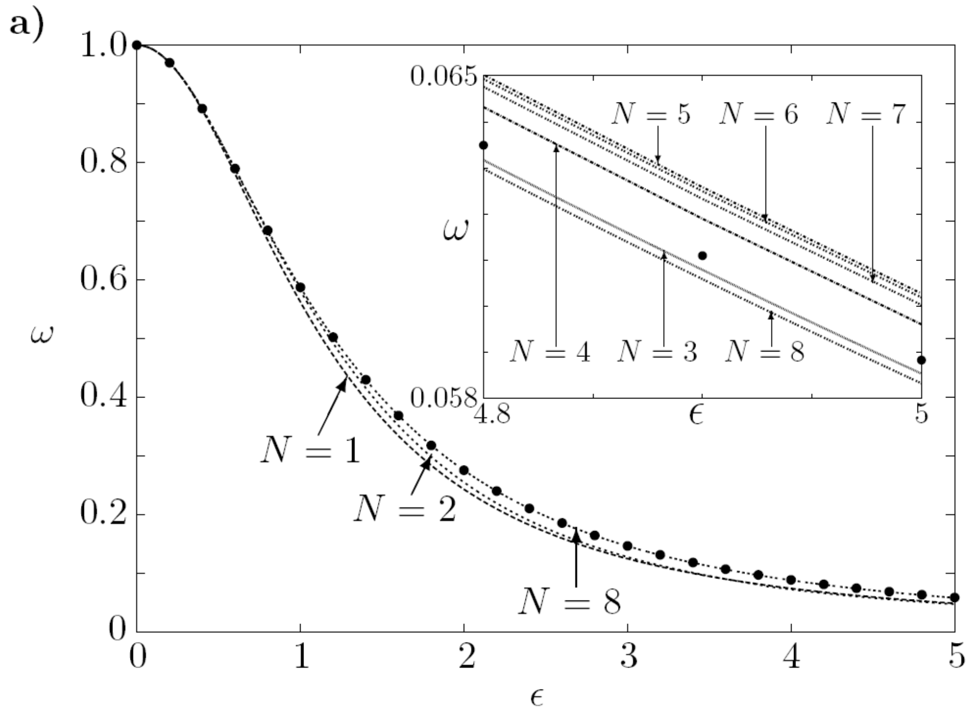
$$\omega^{(N)} = \sum_{n=0}^{N} \omega_{2n}\epsilon^{2n} = \sum_{n=0}^{N} \omega_{2n} g^n \rightarrow \omega_{\mathrm{VPT}}^{(N)} = g^{p/q} \sum_{m=0}^{M} b_m^{(N)} g^{-2m/q}$$

- Introduction of Variational Parameter to the Perturbation Expansion:

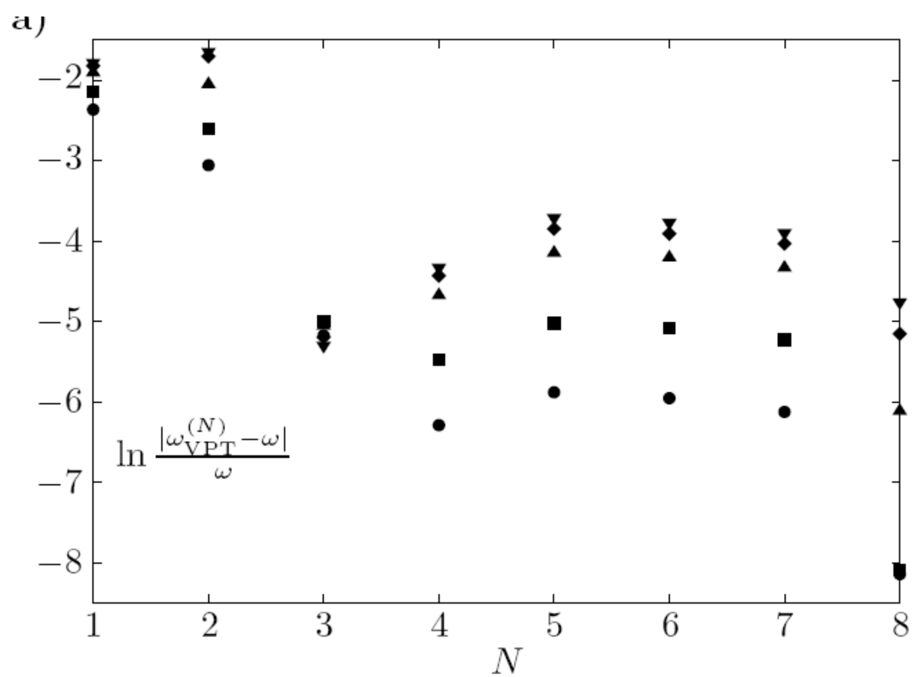
$$\omega_{\mathrm{VPT}}^{(N)}(g, K) = \sum_{n=0}^{N} \omega_{2n} g^n K^{p-nq} \sum_{k=0}^{N-n} \binom{(p-nq)/2}{k} \left(\frac{1}{K^2} - 1\right)^k$$



- **VPT Results:**



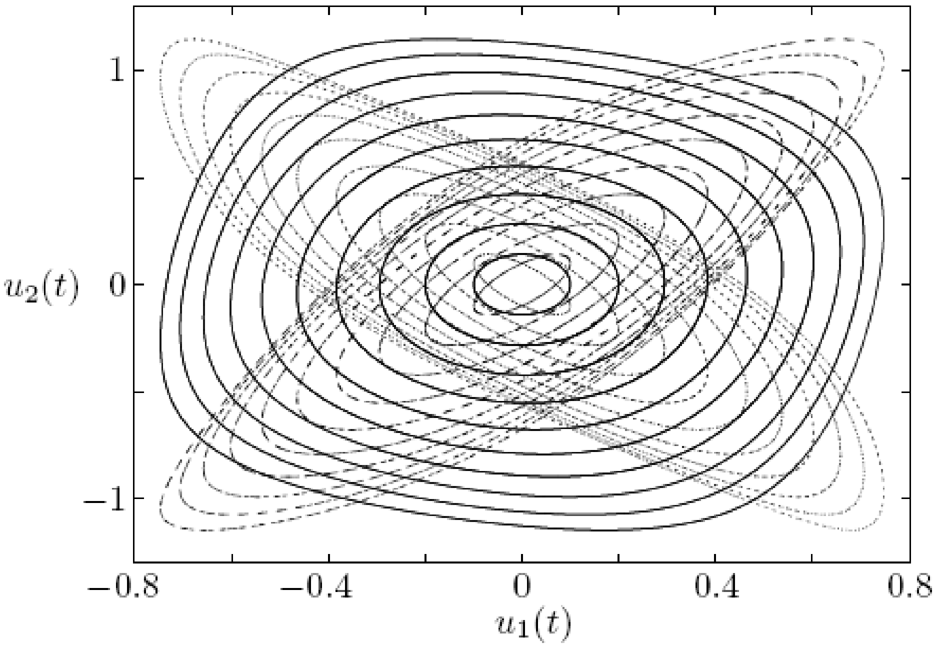
- **Convergence:**

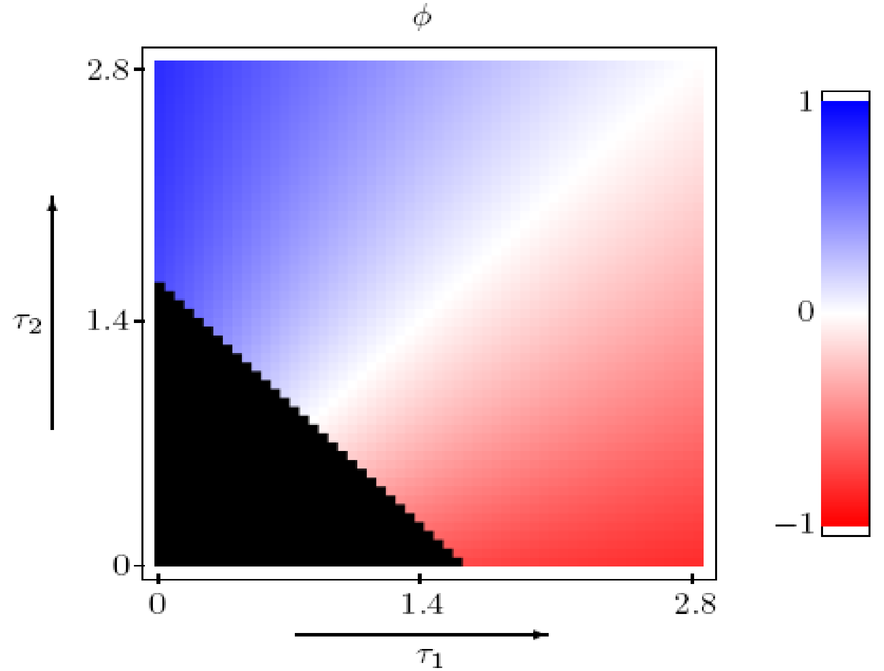


Asymmetric Delays: Numerical Results

- Delay Parameter: $\epsilon = \sqrt{\tau - \tau_0}$

- Covariance: $$\phi = \frac{\int_0^T dt\, u_1(t) u_2(t)}{\left[\int_0^T dt\, u_1(t)u_1(t) \int_0^T dt\, u_2(t)u_2(t)\right]^{1/2}}$$



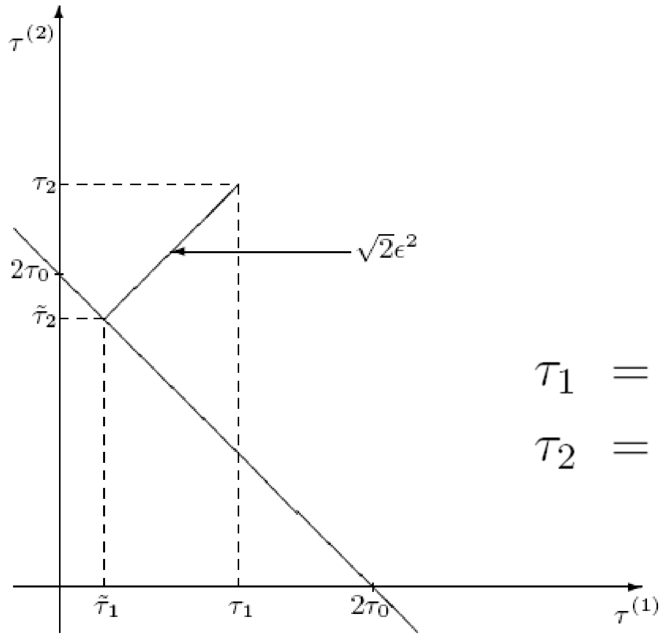


Poincaré-Lindstedt Method

- Expansions: $\epsilon = \sqrt{\tau - \tau_0}$

$$\begin{aligned} \mathbf{u}(t) &= \epsilon \mathbf{U}(t) = \epsilon \left[\mathbf{U}^{(0)}(t) + \epsilon \mathbf{U}^{(1)}(t) + \ldots \right] \\ \omega(\epsilon) &= \omega_0 + \epsilon \omega_1 + \epsilon^2 \omega_2 + \ldots . \end{aligned}$$

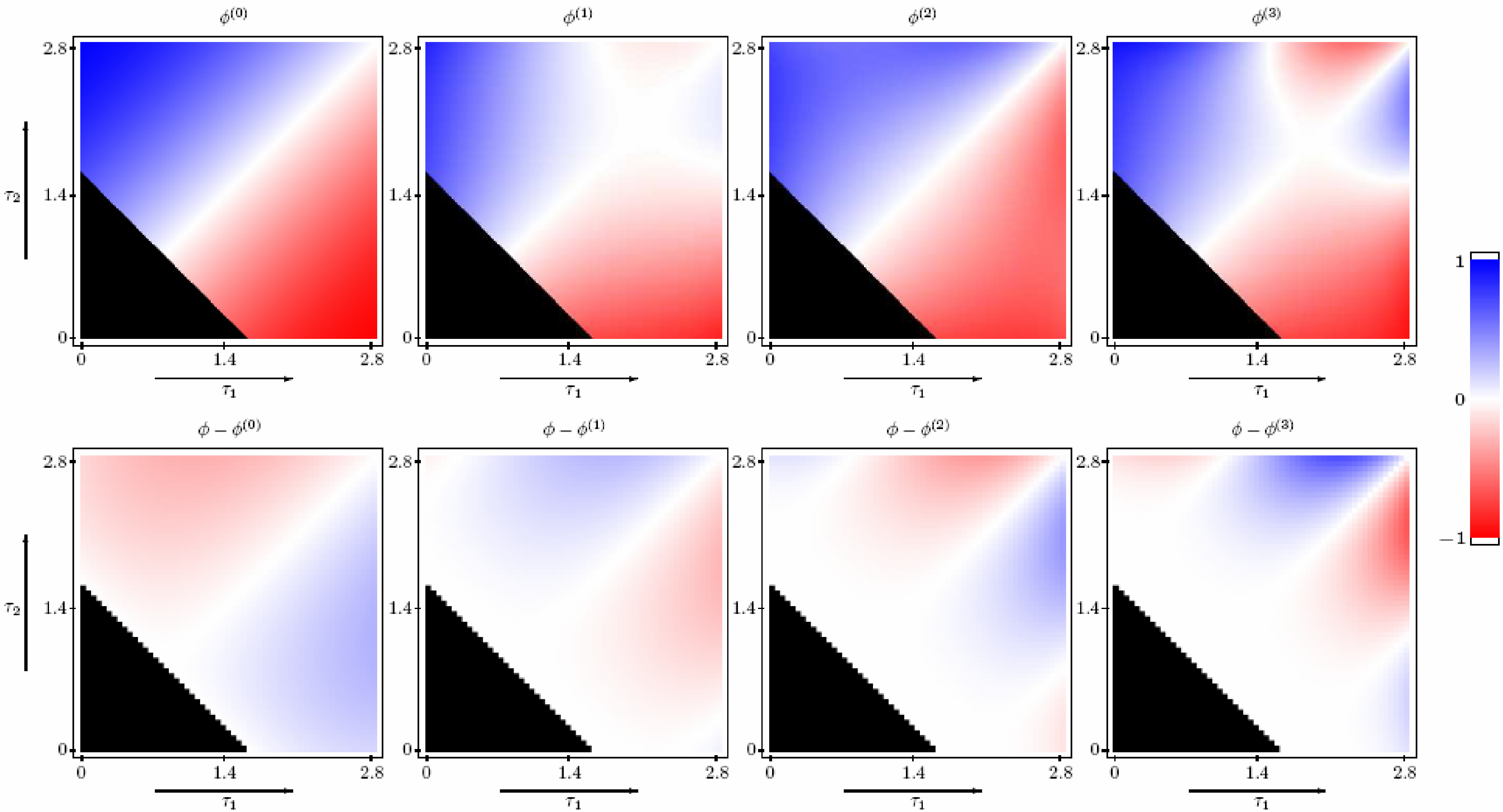
- $\tau^{(1)}$-$\tau^{(2)}$ plane:



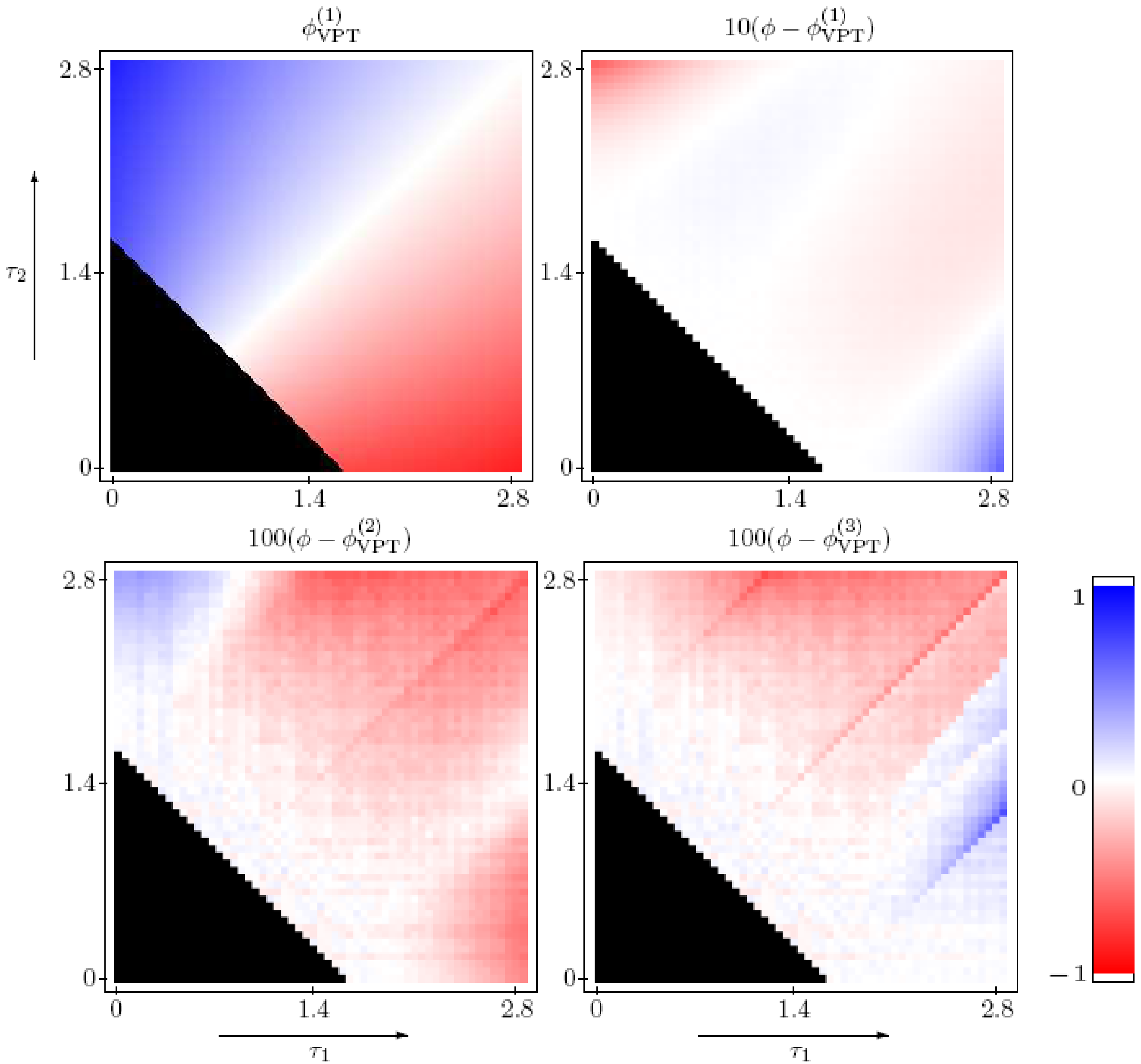
$$\begin{aligned} \tau_1 &= \tilde{\tau}_1 + \epsilon^2 , \\ \tau_2 &= \tilde{\tau}_2 + \epsilon^2 = 2\tau_0 - \tilde{\tau}_1 + \epsilon^2 \end{aligned}$$

- Rescaling: $\xi = \omega(\epsilon) t ,$ $\quad \mathbf{U}(t) = \mathbf{V}(\xi)$

Perturbative Results



VPT Results



$\phi^{(1)}_{\mathrm{VPT}}$
$10(\phi - \phi^{(1)}_{\mathrm{VPT}})$
$100(\phi - \phi^{(2)}_{\mathrm{VPT}})$
$100(\phi - \phi^{(3)}_{\mathrm{VPT}})$
τ_2
τ_1
0
1.4
2.8
1
0
−1

Summary

- In Neural Networks, Oscillations Can Arise from Temporal Delays
- In a Two-Neuron Network, Asymmetric Delays Can Control the Phase of the Oscillations and lead to Synchrony
- VPT Permits the Evaluation of Divergent Series for Strong Coupling
- The Perturbation Series for the Frequency and the Phase of the Oscillations as well as the Fourier Series for the Limit Cycle Can be Resummed Efficiently Using VPT

$$f^{(N)}(g) = \sum_{n=0}^{N} a_n g^n \xrightarrow{\text{VPT}} f(g) = g^{p/q} \sum_{m=0}^{M} b_m g^{-2m/q}$$