Stability of a Dilute Ultracold Trapped Gas of Bose and Fermi Atoms

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Overview

- 1. Grand-Canonical Ensemble
- 2. Density Profiles

  G. Roati, F. Riboli, G. Modugno and M. Inguscio, *Fermi-Bose Quantum Degenerate ^{40}K-^{87}Rb Mixture with Attractive Interaction*, Phys. Rev. Lett. **89**, 15 (2002)]

  C. Ospelkaus, S. Ospelkaus, K. Sengstock and K. Bongs, *Interaction-driven Dynamics of ^{40}K / ^{87}Rb Fermi-Bose Gas Mixture in the Large Particle Number Limit*, arXiv:cond-matt/0507219v2 7 Sep 2005
- 3. Stability
- 4. Phase Diagramms
- 5. Outlook

1.1. Partition Function

Definition:

$$\mathcal{Z} = \oint \mathcal{D}\psi_B^* \oint \mathcal{D}\psi_B \oint \mathcal{D}\psi_F^* \oint \mathcal{D}\psi_F \, e^{-\mathcal{A}[\psi_B^*,\psi_B,\psi_F^*,\psi_F]/\hbar}$$

Euclidean Action:

$$\begin{aligned}\mathcal{A}[\psi_B^*,\psi_B,\psi_F^*,\psi_F] &= \mathcal{A}_B[\psi_B^*,\psi_B] + \mathcal{A}_F[\psi_F^*,\psi_F] \\ &\quad + \mathcal{A}_{BF}[\psi_B^*,\psi_B,\psi_F^*,\psi_F]\end{aligned}$$

Contact Interaction: $V_{ij}^{(\mathrm{int})}(\mathbf{x},\mathbf{x}') = g_{ij}\,\delta(\mathbf{x}-\mathbf{x}'), \quad i,j = B,F$

Bose Action:

$$\mathcal{A}_B[\psi_B^*,\psi_B] = \int_0^{\hbar\beta} d\tau \int d^D x \, \psi_B^*(\mathbf{x},\tau) \left[\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2}{2m_B}\Delta \right.$$
$$\left. + V_B(\mathbf{x}) - \mu_B + \frac{1}{2}\int d^D x' \, V_{BB}^{(\mathrm{int})}(\mathbf{x},\mathbf{x}') \, |\psi_B(\mathbf{x}',\tau)|^2 \right] \psi_B(\mathbf{x},\tau)$$

Fermi Action:

$$\mathcal{A}_F[\psi_F^*,\psi_F] = \int_0^{\hbar\beta} d\tau \int d^D x \, \psi_F^*(\mathbf{x},\tau) \left[\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2}{2m_F}\Delta \right.$$
$$\left. - V_F(\mathbf{x}) - \mu_F \right] \psi_F(\mathbf{x},\tau)$$

Interaction:

$$\mathcal{A}_{BF}[\psi_B^*,\psi_B,\psi_F^*,\psi_F] = \int_0^{\hbar\beta} d\tau \int d^D x \int d^D x' \, V_{BF}^{(\mathrm{int})}(\mathbf{x},\mathbf{x}')$$
$$\times \, |\psi_B(\mathbf{x}',\tau)|^2 \, |\psi_F(\mathbf{x},\tau)|^2$$

1.2. Background Method

Split of the Bosonic Field:

$$\psi_B(\mathbf{x},\tau) = \Psi_0(\mathbf{x}) + \delta\psi_B(\mathbf{x},\tau)$$

Taylor Expansion:

$$\mathcal{A}[\Psi_0^* + \delta\psi_B^*, \Psi_0 + \delta\psi_B, \psi_F^*, \psi_F] = \mathcal{A}^{(0)}[\delta\psi_B^*, \delta\psi_B] + \mathcal{A}^{(2)}[\delta\psi_B^*, \delta\psi_B] + \mathcal{A}^{(\mathrm{int})}[\delta\psi_B^*, \delta\psi_B]$$

Partition Function in the Zeroth Order of $\delta\psi_B^*(\mathbf{x},\tau), \delta\psi_B(\mathbf{x},\tau)$:

$$\mathcal{Z}[\Psi_0^*, \Psi_0] = e^{-\mathcal{A}_B[\Psi_0^*,\Psi_0]/\hbar}\, \mathcal{Z}_F[\Psi_0^*, \Psi_0]$$

Functional Integral over Fermionic Fields:

$$\mathcal{Z}_F[\Psi_0^*, \Psi_0] = \oint \mathcal{D}\psi_F^* \oint \mathcal{D}\psi_F\, e^{-\mathcal{A}_F[\psi_F^*,\psi_F]/\hbar} \times e^{-\mathcal{A}_{BF}[\psi_B^*,\psi_B,\Psi_F^*,\Psi_F]/\hbar}$$

Fermi Action and Interaction:

$$\mathcal{A}_F[\psi_F^*, \psi_F] + \mathcal{A}_{BF}[\Psi_0^*, \Psi_0, \psi_F^*, \psi_F] = \int_0^{\hbar\beta} d\tau \int d^D x \times \psi_F^*(\mathbf{x},\tau) \left[\hbar \frac{\partial}{\partial\tau} + \hat{H}_F(\mathbf{x}) - \mu_F\right] \psi_F(\mathbf{x},\tau)$$

Effective Hamiltonian for Fermions:

$$\hat{H}_F(\mathbf{x}) = -\frac{\hbar^2}{2m_F}\Delta + V_F(\mathbf{x}) + \int d^D x'\, V_{BF}^{(\mathrm{int})}(\mathbf{x},\mathbf{x}')\, |\Psi_0(\mathbf{x}')|^2$$

1.3. Fermionic Functional Integral

Decomposition of the Fermionic Field:

$$\psi_F(\mathbf{x}, \tau) = \sum_{\mathbf{n}} \sum_{m=-\infty}^{\infty} c_{\mathbf{n}\,m}\, \psi_{\mathbf{n}}(\mathbf{x})\, e^{-i\omega_m \tau}$$

Eigenvalue Problem of the One-Particle wavefunction:

$$\hat{H}_F(\mathbf{x})\, \psi_{\mathbf{n}}(\mathbf{x}) = E_{\mathbf{n}}\, \psi_{\mathbf{n}}(\mathbf{x})$$

Matsubara Frequencies:

$$\omega_m = \frac{2\pi}{\hbar\beta} m, \qquad\qquad m = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2} \ldots$$

Result of the Fermionic Functional Integral:

$$\mathcal{Z}_F[\Psi_0^*, \Psi_0] = \prod_{\mathbf{n}} \prod_{m=-\infty}^{\infty} \left(-i\hbar\omega_m + E_{\mathbf{n}} - \mu_F\right)$$

Grand-Canonical Free Energy $\mathcal{F} \equiv -\ln \mathcal{Z}/\beta$:

$$\mathcal{F}[\Psi_0^*, \Psi_0] = \frac{1}{\hbar\beta} \mathcal{A}_B[\Psi_0^*, \Psi_0] \\ -\frac{1}{\beta} \sum_{\mathbf{n}} \ln\left[2\cosh\left(\frac{\beta(E_{\mathbf{n}} - \mu_F)}{2}\right)\right]$$

1.4. Semiclassical Approximation $(\Delta E \ll k_B T)$

Eigenvalues:

$$E_{\mathbf{p}}(\mathbf{x}) = \frac{\mathbf{p}^2}{2m_F} + V_F(\mathbf{x}) + \int d^D x' \, V_{BF}^{(\mathrm{int})}(\mathbf{x}, \mathbf{x}') \, |\Psi_0(\mathbf{x}')|^2$$

Replacement:

$$\sum_{\mathbf{n}} \to \int d^D x \int \frac{d^D p}{(2\pi\hbar)^D}$$

Free Energy:

$$\mathcal{F}[\Psi_0^*, \Psi_0] = \frac{1}{\hbar\beta} \mathcal{A}_B[\Psi_0^*, \Psi_0] \\ -\kappa \int d^3 x \int_0^\infty d\varepsilon \, \frac{\varepsilon^{3/2}}{e^{\beta[\varepsilon - \tilde{\mu}_F(\mathbf{x})]} + 1}$$

Abbreviations:

$$\tilde{\mu}_F(\mathbf{x}) \equiv \mu_F - V_F(\mathbf{x}) - \int d^D x' \, V_{BF}^{(\mathrm{int})}(\mathbf{x}, \mathbf{x}') \, |\Psi_0(\mathbf{x}')|^2,$$

$$\kappa \equiv \frac{2^{1/2} m_F^{3/2}}{3\pi^2 \hbar^3}$$

Sommerfeld Expansion in the Zero-Temperature Limit $T \downarrow 0$:

$$\mathcal{F}[\Psi_0^*, \Psi_0] = \frac{1}{\hbar\beta} \mathcal{A}_B[\Psi_0^*, \Psi_0] \\ -\frac{2\kappa}{5} \int d^3 x \, \Theta(\tilde{\mu}_F(\mathbf{x})) \, \tilde{\mu}_F(\mathbf{x})^{5/2}$$

1.5. Extremization of the Effective Action

Gross-Pitaevskii Equation:

$$\left[-\frac{\hbar^2}{2m_B}\Delta + V_B(\mathbf{x}) - \mu_B + \int d^3x'\, V_{BB}^{(\mathrm{int})}(\mathbf{x},\mathbf{x}')\, |\Psi_0(\mathbf{x}')|^2 \right.$$
$$\left. + \kappa \int d^3x''\, \Theta(\tilde{\mu}_F(\mathbf{x}''))\, V_{BF}^{(\mathrm{int})}(\mathbf{x}'',\mathbf{x})\, \tilde{\mu}_F(\mathbf{x}'')^{3/2}\right] \Psi_0(\mathbf{x}) = 0$$

Normalisation:

$$N_B = -\frac{\partial \mathcal{F}}{\partial \mu_B} = \int d^3x\, |\Psi_0(\mathbf{x})|^2,$$
$$N_F = -\frac{\partial \mathcal{F}}{\partial \mu_F} = \kappa \int d^3x\, \Theta(\tilde{\mu}_F(\mathbf{x}))\, \tilde{\mu}_F(\mathbf{x})^{3/2}$$

Particle Densities:

$$n_B(\mathbf{x}) = |\Psi_0(\mathbf{x})|^2,$$
$$n_F(\mathbf{x}) = \kappa\, \Theta(\tilde{\mu}_F(\mathbf{x}))\, \tilde{\mu}_F(\mathbf{x})^{3/2}$$

Thomas-Fermi Approximation $-\hbar^2/(2m_F)\Delta \to 0$:

$$V_B(\mathbf{x}) - \mu_B + \int d^3x'\, V_{BB}^{(\mathrm{int})}(\mathbf{x},\mathbf{x}')\, n_B(\mathbf{x}')$$
$$+ \int d^3x''\, V_{BF}^{(\mathrm{int})}(\mathbf{x}'',\mathbf{x})\, n_F(\mathbf{x}'') = 0$$

1.6. Specialization

3-dim. Anisotropic Harmonic Trap:

$$V_i(\mathbf{x}) = \frac{m_i}{2}\left(\omega_{i,r}^2 r^2 + \omega_{i,z}^2 z^2\right), \qquad i = B, F$$

Short-Range Contact Interaction:

$$V_{ij}^{(\mathrm{int})}(\mathbf{x}, \mathbf{x}') = g_{ij}\,\delta(\mathbf{x} - \mathbf{x}'), \qquad i, j = B, F$$

$$g_{ij} = 2\pi\hbar^2 a_{ij}\,\frac{m_i + m_j}{m_i\,m_j}$$

Algebraic GP Equation:

$$V_B(\mathbf{x}) - \mu_B + g_{BB}\,n_B(\mathbf{x}) + g_{BF}\,n_F(\mathbf{x}) = 0$$

Particle Density of Fermions:

$$\begin{aligned} n_F(\mathbf{x}) = \kappa\,\Theta\left(\mu_F - V_F(\mathbf{x}) - g_{BF}\,n_B(\mathbf{x})\right) \\ \times \left[\mu_F - V_F(\mathbf{x}) - g_{BF}\,n_B(\mathbf{x})\right]^{3/2} \end{aligned}$$

1.7. Dimensionless Form of GP Equation:

Division by $V_B(\tilde{R}_B) \equiv m_B \tilde{\omega}_B^2 \tilde{R}_B^2/2$:

$$X_r^2 + X_z^2 - M_B + N_B(\mathbf{X}) - N_F(\mathbf{X}) = 0$$

Relative Coordinates and Chemical Potentials:

$$X_k \equiv \frac{k}{R_{B,k}}, \quad k = r, z, \qquad M_i \equiv \frac{\mu_i}{V_B(\tilde{R}_B)}, \quad i = B, F$$

Thomas-Fermi-Radii in the limit $a_{BF} \to 0$:

$$R_{B,k} = \sqrt{\frac{2\mu_B}{m_B\, \omega_{B,k}^2}}$$

Relative Density of Bosons:

$$N_B(\mathbf{X}) \equiv \frac{g_{BB}}{V_B(\tilde{R}_B)}\, n_B(\mathbf{x})$$

Relative Density of Fermions:

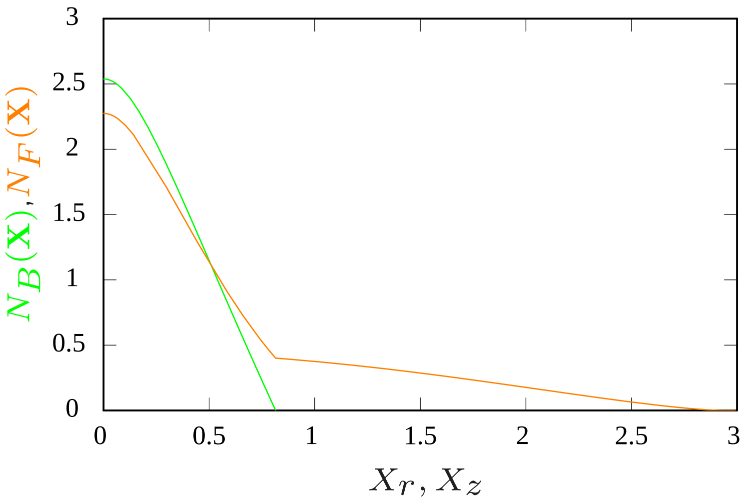
$$N_F(\mathbf{X}) \equiv -\frac{g_{BF}}{V_B(\tilde{R}_B)}\, n_F(\mathbf{x})$$
$$= \kappa\, g_{BF} \sqrt{V_B(\tilde{R}_B)}\, \Theta\left(M_F - X_r^2 + X_z^2 - \frac{g_{BF}}{g_{BB}}\, N_B(\mathbf{x})\right)$$
$$\times \left[M_F - X_r^2 + X_z^2 - \frac{g_{BF}}{g_{BB}}\, N_B(\mathbf{x})\right]^{3/2}$$

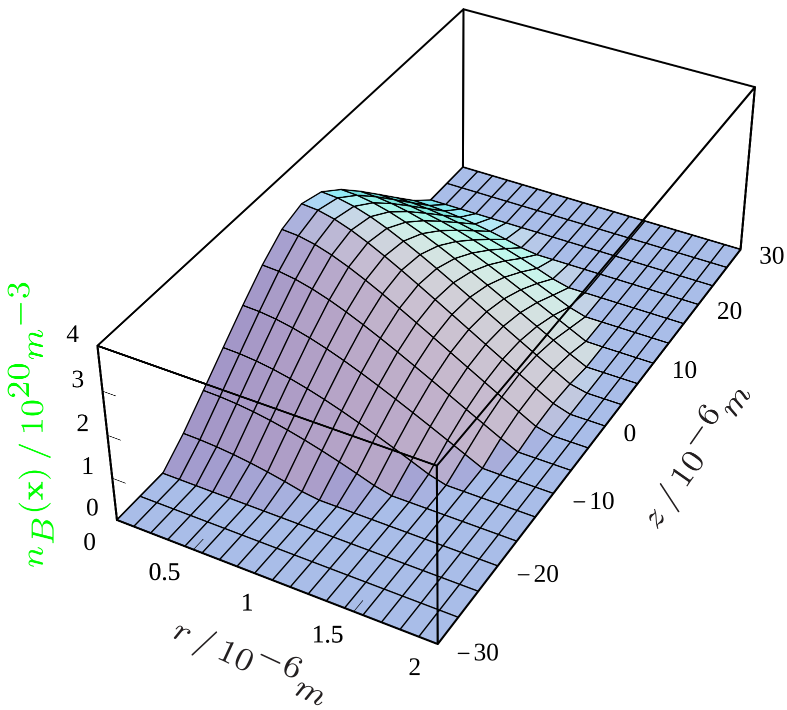
2. Density Profiles for a ^{87}Rb-^{40}K-Mixture

Florence Experiment

Experimental Parameters:
$\omega_{B,r} = 2\pi \times 215Hz$
$\omega_{B,z} = 2\pi \times 16.3Hz$
$\omega_F = \sqrt{m_B/m_F}\omega_B \approx 1.47\omega_B$
$a_{BB} = 5.25nm$
$a_{BF} = -21.7nm$

Example:
$N_B = 24800 \quad N_F = 5800$

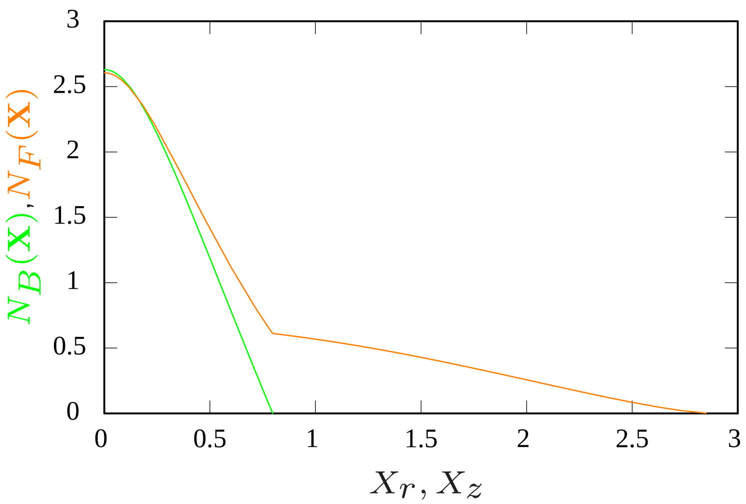


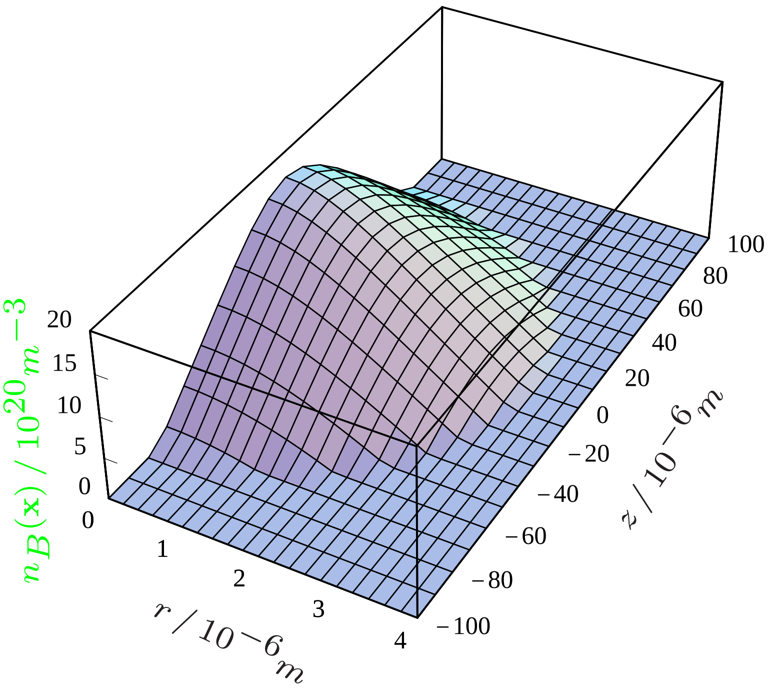


Hamburg Experiment

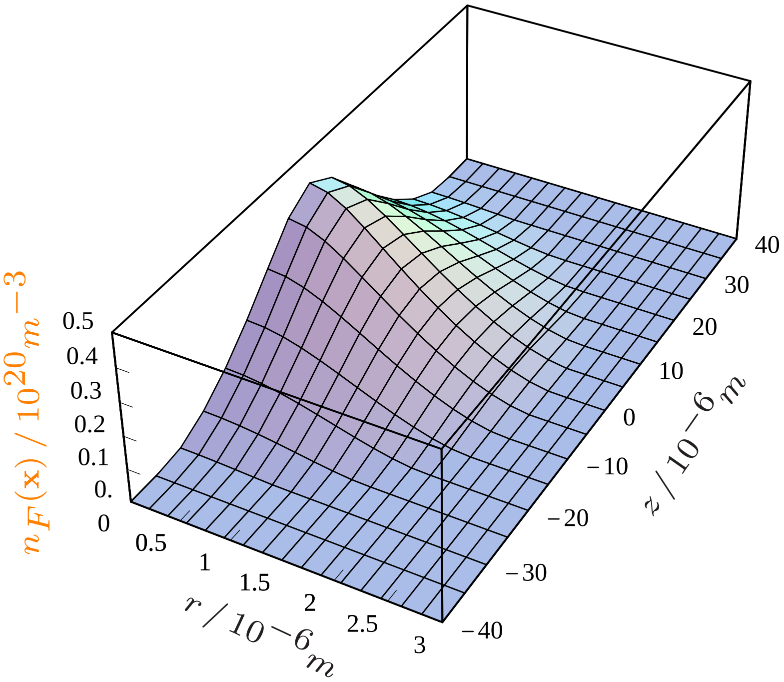
Experimental Parameters:
$\omega_{B,r} = 2\pi \times 257Hz$
$\omega_{B,z} = 2\pi \times 11.3Hz$
$\omega_F = \sqrt{m_B/m_F}\omega_B \approx 1.47\omega_B$
$a_{BB} = 5.238nm$
$a_{BF} = -14.87nm$

Example:
$N_B = 8.6 \cdot 10^6 \quad N_F = 831000$

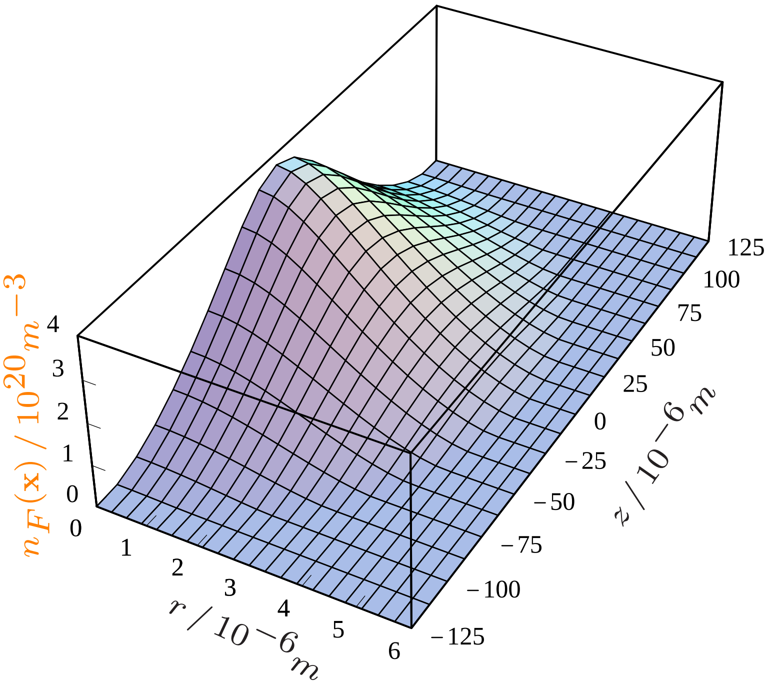




Florence Experiment



Hamburg Experiment



3. Stability

Grand-Canonical Free Energy:

$$\mathcal{F}[\Psi_0^*, \Psi_0] = \int d^3x \left[\frac{\hbar^2}{2m_B} |\nabla \Psi_0(\mathbf{x})|^2 + (V_B(\mathbf{x}) - \mu_B) |\Psi_0(\mathbf{x})|^2 + \frac{1}{2} g_{BB} |\Psi_0(\mathbf{x})|^4 - \frac{2\kappa}{5} \Theta(\tilde{\mu}_F(\mathbf{x}))\, \tilde{\mu}_F(\mathbf{x})^{5/2} \right]$$

Effective Chemical Potential:

$$\tilde{\mu}_F(\mathbf{x}) \equiv \mu_F - V_F(\mathbf{x}) - g_{BF} |\Psi_0(\mathbf{x})|^2$$

Ansatz with Variational Width w:

$$\Psi_0(\mathbf{x}) = \sqrt{\frac{N_B \lambda}{\pi^{3/2} w^3}} \exp\left\{ -\frac{r^2 + \lambda^2 z^2}{2w^2} \right\}, \qquad \lambda = \sqrt{\frac{\omega_z}{\omega_r}}$$

Condition for Stability Limit of the Bose Subsystem:

$$N_B = N_{Bcrit} \quad \Leftrightarrow \quad \left. \frac{\partial \mathcal{F}(w)}{\partial w} \right|_{w=w_{crit}} = \left. \frac{\partial^2 \mathcal{F}(w)}{\partial w^2} \right|_{w=w_{crit}} = 0$$

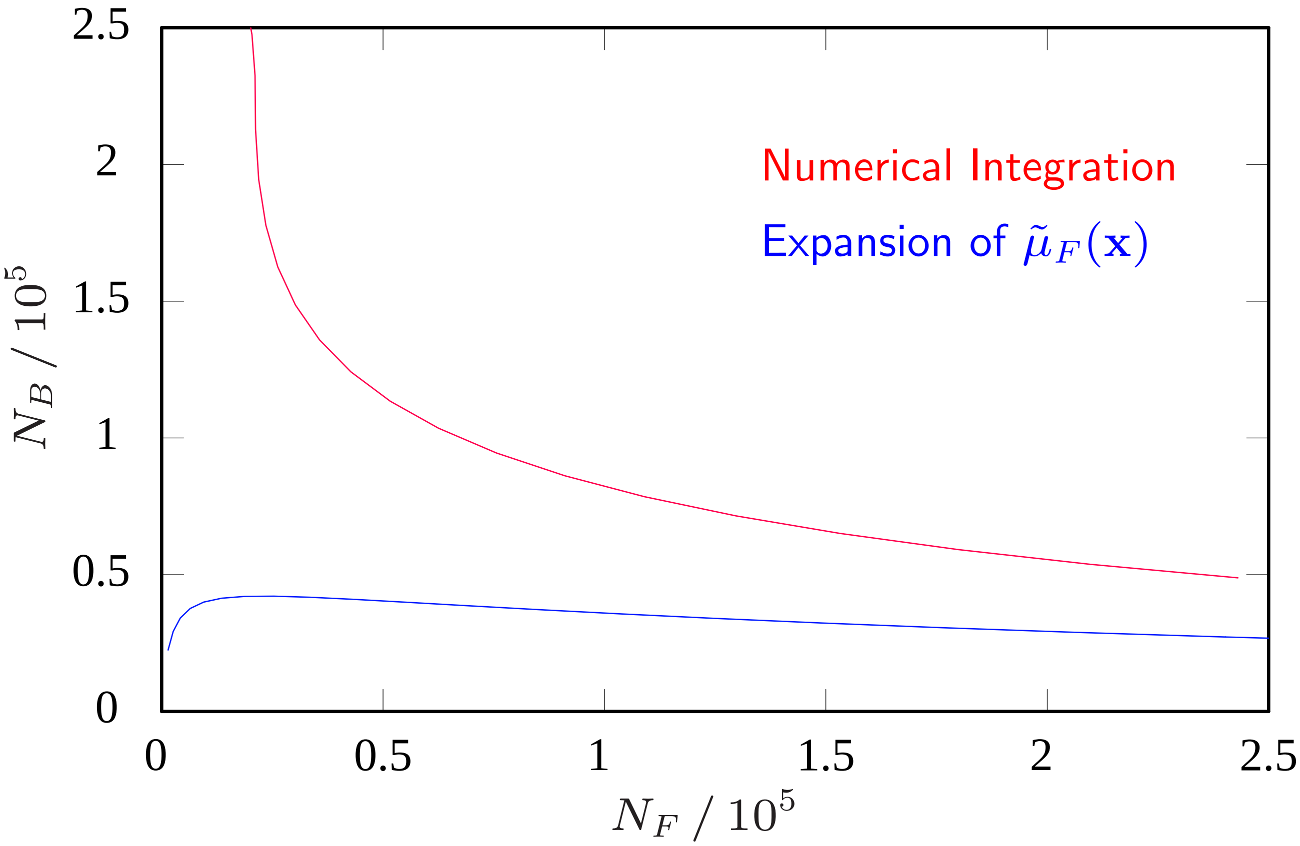
Expansion of $\tilde{\mu}_F(\mathbf{x})$ up to the $3_{\rm rd}$ order in g_{BF}:

S.-T. Chui, V. N. Ryzhov and E. E. Tareyeva, *Collapse of Bose component in Bose-Fermi mixture with atraction between components*, arXiv:cond-matt/0412701 v1 27 Dec 2004

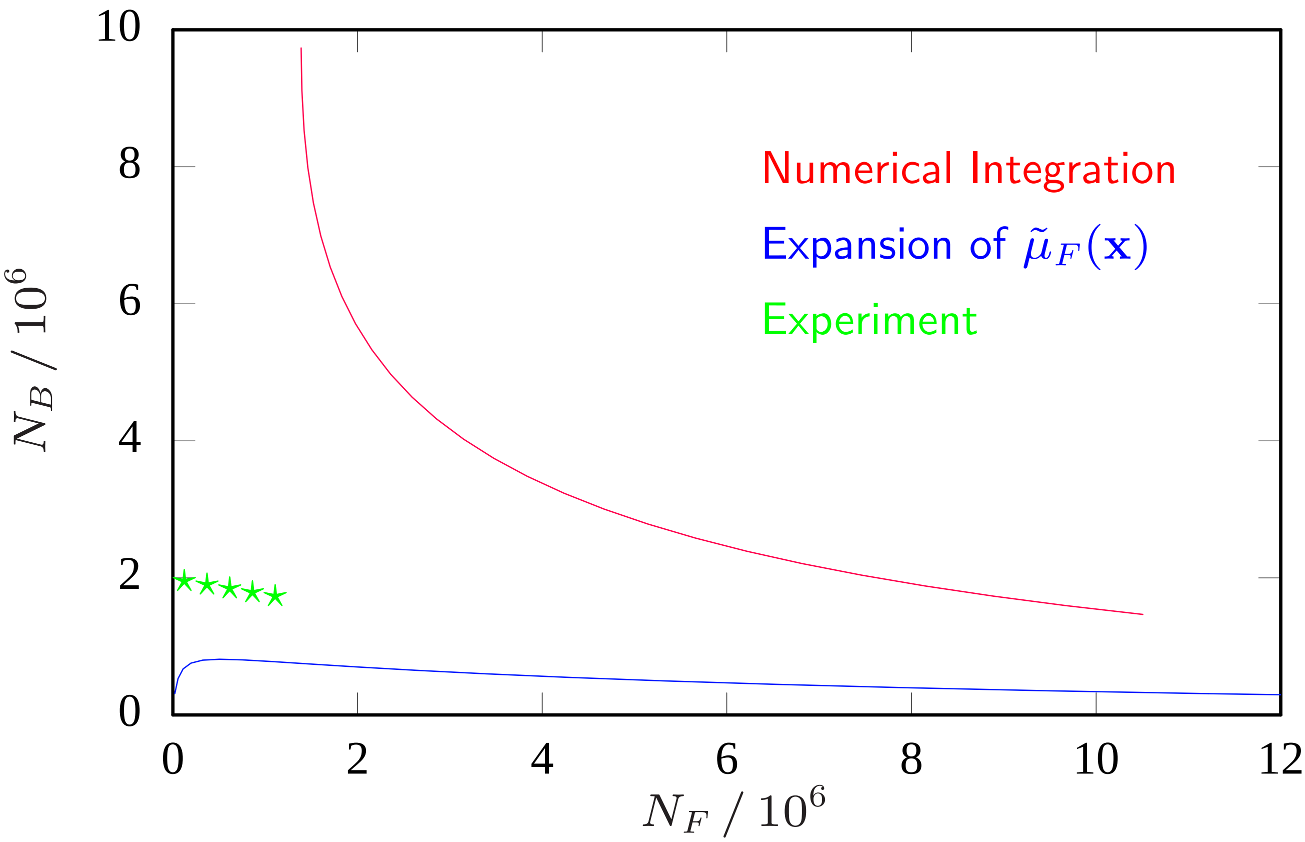
$$\mathcal{F}_{ex}[\Psi_0^*, \Psi_0] = \int d^3x \left[\frac{\hbar^2}{2m_B} |\nabla \Psi_0(\mathbf{x})|^2 + (V_B(\mathbf{x}) - \mu_B) |\Psi_0(\mathbf{x})|^2 + \frac{g_{eff}}{2} |\Psi_0(\mathbf{x})|^4 + \frac{\kappa g_{BF}^3}{8\mu_F^{1/2}} |\Psi_0(\mathbf{x})|^6 \right]$$

4. Phase Diagrams

Florence Experiment:



Hamburg Experiment:



4. Outlook

- Discrepancy in the Phase Diagrams between both Methods
- Variational Approach with two Widths w_r and w_z
- ^{52}Cr-^{53}Cr-Mixture with Dipol-Dipol Interaction
  (Stuttgart Experiment - T. Pfau)