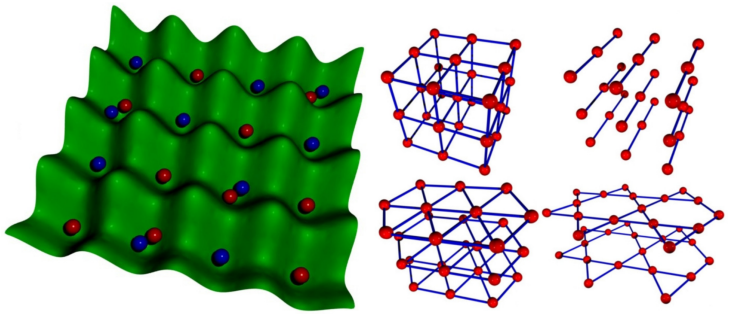
Ginzburg-Landau Theory for Bosonic Gases in Optical Lattices

F. E. A. dos Santos

Freie Universität Berlin
Fachbereich Physik

PhD Exam - Berlin, November 7, 2011



Freie Universität Berlin

Outline

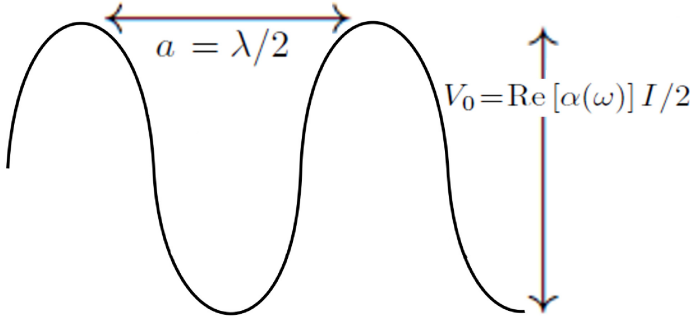
Outline

Optical Lattices

- Counter-propagating laser beams create periodic potential

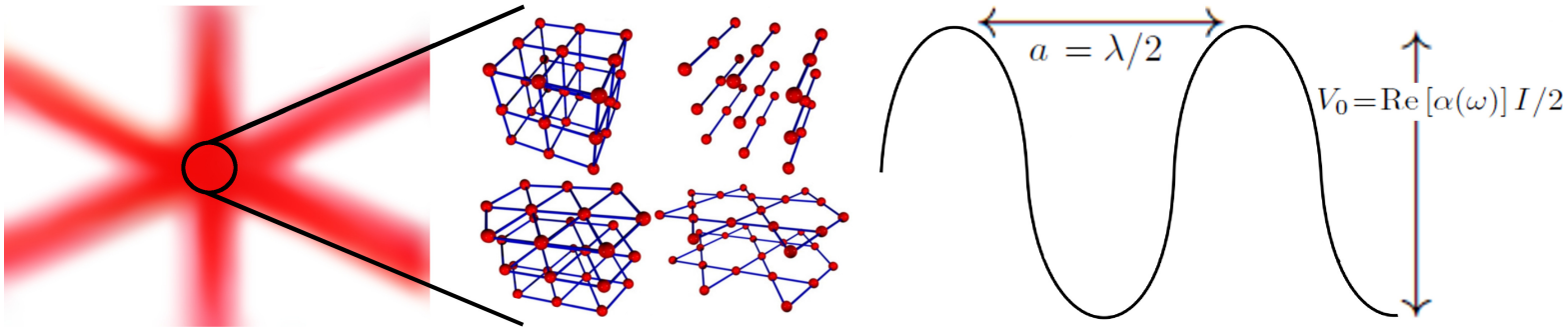
Optical Lattices

- Counter-propagating laser beams create periodic potential
- Hopping and interactions are highly controllable



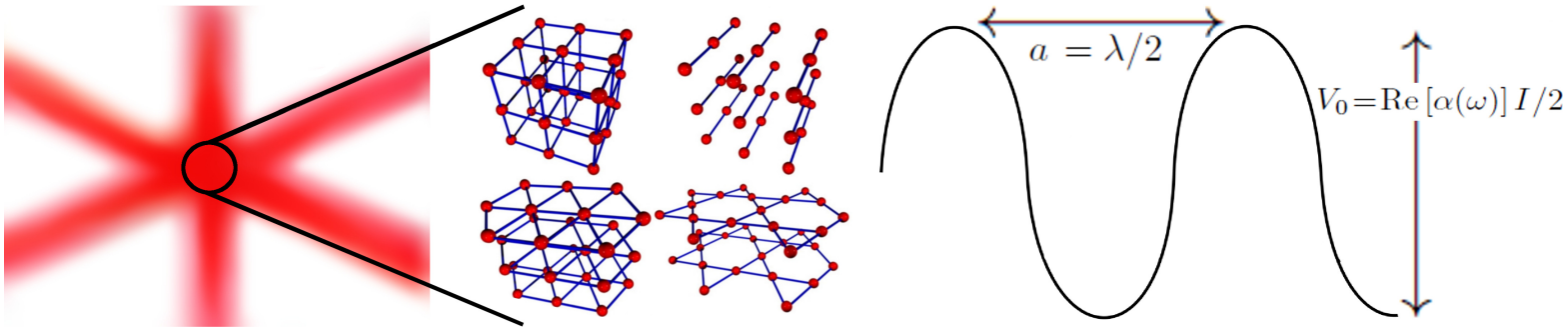
Optical Lattices

- Counter-propagating laser beams create periodic potential
- Hopping and interactions are highly controllable
- Different possible topologies at 1D, 2D, and 3D



Optical Lattices

- Counter-propagating laser beams create periodic potential
- Hopping and interactions are highly controllable
- Different possible topologies at 1D, 2D, and 3D
- Model for condensate matter systems

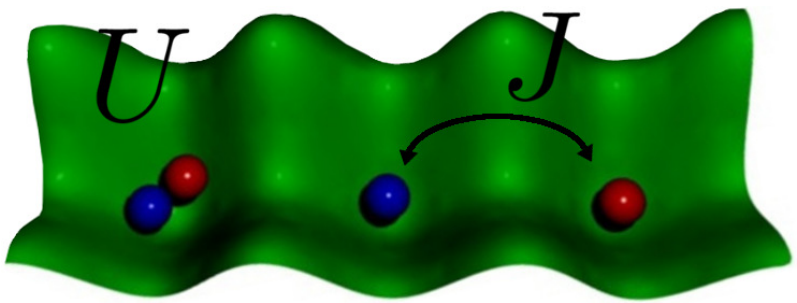


Outline

Bose-Hubbard Model

- Bose-Hubbard Hamiltonian

$$\hat{H}_{\mathrm{BH}} = \hat{H}_0 + \hat{H}_J$$



$$\hat{H}_0 = \sum_i \frac{U}{2}(\hat{n}_i^2 - \hat{n}_i) - \mu \hat{n}_i$$

$$\hat{H}_J = -\sum_{ij} J_{ij} \hat{a}_i^\dagger \hat{a}_j$$

$$\hat{n}_i = \hat{a}_i^\dagger \hat{a}_i$$

$$J_{ij} = \begin{cases} J, & \text{if } i,j \text{ nearest neighbors} \\ 0, & \text{otherwise.} \end{cases}$$

Bose-Hubbard Model

- Bose-Hubbard Hamiltonian

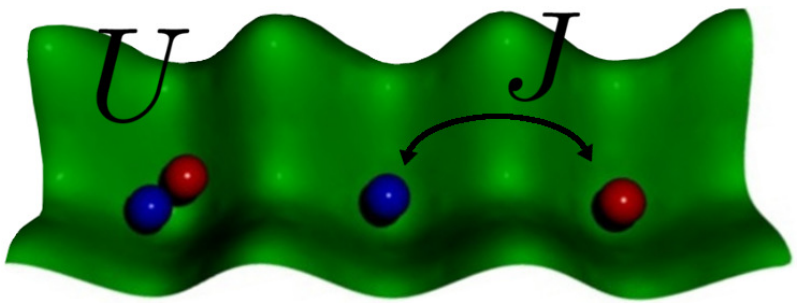
$$\hat{H}_{\mathrm{BH}} = \hat{H}_0 + \hat{H}_J$$

$$\hat{H}_0 = \sum_i \frac{U}{2}(\hat{n}_i^2 - \hat{n}_i) - \mu \hat{n}_i$$

$$\hat{H}_J = -\sum_{ij} J_{ij} \hat{a}_i^\dagger \hat{a}_j$$

$$\hat{n}_i = \hat{a}_i^\dagger \hat{a}_i$$

$$J_{ij} = \begin{cases} J, & \text{if } i,j \text{ nearest neighbors} \\ 0, & \text{otherwise.} \end{cases}$$



- $\hat{H}_0$ is diagonal

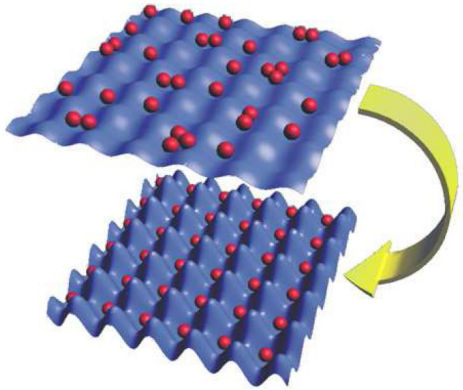
$$\hat{H}_0 |n\rangle = N_S E_n |n\rangle$$
$$E_n = \frac{U}{2} n(n-1) - \mu n$$

- Expansion in series of J

Outline

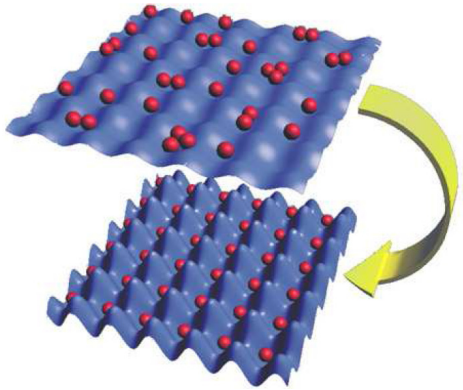
Superfluid-Mott Insulator Transition

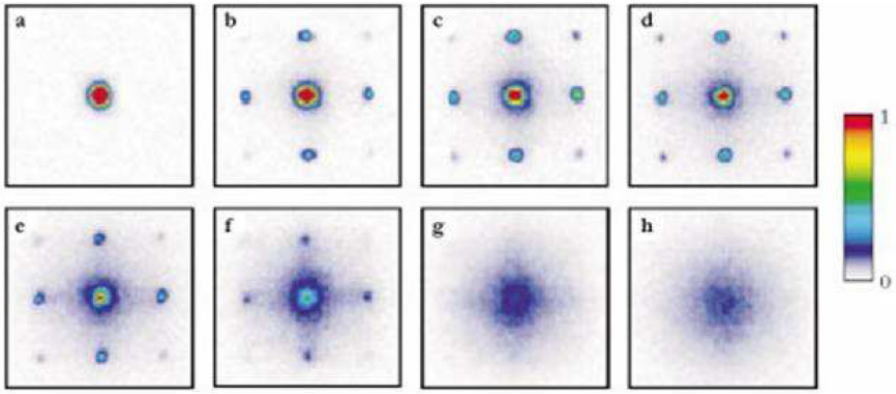
- Increasing the laser intensity localizes atoms



Superfluid-Mott Insulator Transition

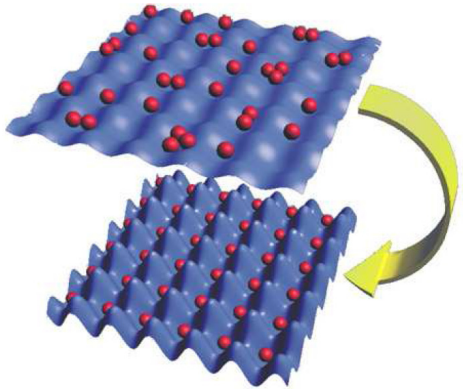
- Increasing the laser intensity localizes atoms
- Detectable in time-of-flight pictures

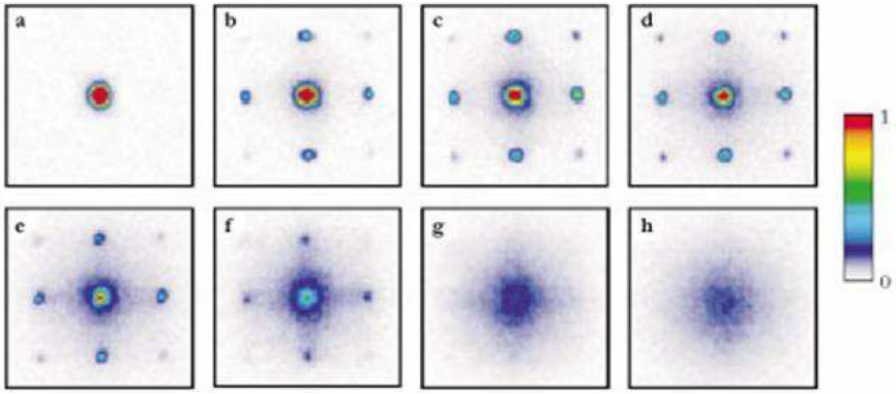


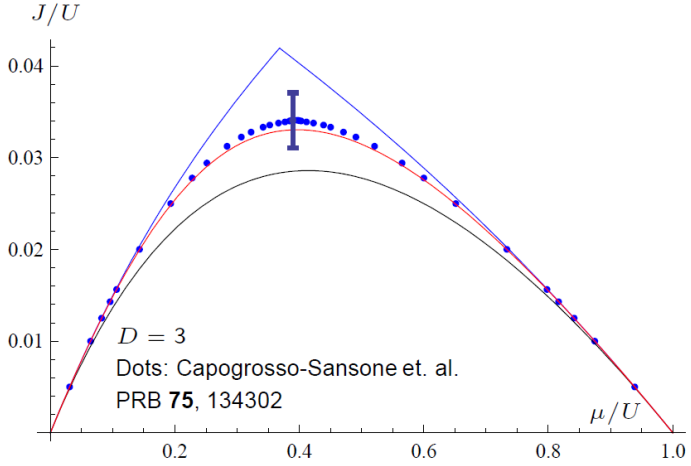


Superfluid-Mott Insulator Transition

- Increasing the laser intensity localizes atoms
- Detectable in time-of-flight pictures
- Inaccurate analytical methods prior to this work







Outline

Generating Functionals

- Symmetry breaking source:

$$\hat{H}_{\mathrm{BH}}(\tau) = \hat{H}_{\mathrm{BH}} + \sum_i \left[j_i^*(\tau)\hat{a}_i + j_i(\tau)\hat{a}_i^\dagger \right]$$

Generating Functionals

- Symmetry breaking source:

$$\hat{H}_{\mathrm{BH}}(\tau) = \hat{H}_{\mathrm{BH}} + \sum_i \left[j_i^*(\tau)\hat{a}_i + j_i(\tau)\hat{a}_i^\dagger \right]$$

- Evolution operator: $\hat{U}(\tau, \tau_0) = \hat{T} e^{-\int_{\tau_0}^{\tau} d\tau' \hat{H}_{\mathrm{BH}}(\tau)}$

Generating Functionals

- Symmetry breaking source:

$$\hat{H}_{\mathrm{BH}}(\tau) = \hat{H}_{\mathrm{BH}} + \sum_i \left[j_i^*(\tau)\hat{a}_i + j_i(\tau)\hat{a}_i^\dagger \right]$$

- Evolution operator: $\hat{U}(\tau, \tau_0) = \hat{T} e^{-\int_{\tau_0}^{\tau} d\tau' \hat{H}_{\mathrm{BH}}(\tau)}$
- Generalized partition function: $Z = \mathrm{Tr}\left[\hat{U}(0, \beta)\right]$

Generating Functionals

- Symmetry breaking source:

$$\hat{H}_{\mathrm{BH}}(\tau) = \hat{H}_{\mathrm{BH}} + \sum_i \left[j_i^*(\tau)\hat{a}_i + j_i(\tau)\hat{a}_i^\dagger \right]$$

- Evolution operator: $\hat{U}(\tau,\tau_0) = \hat{T} e^{-\int_{\tau_0}^{\tau} d\tau' \hat{H}_{\mathrm{BH}}(\tau)}$
- Generalized partition function: $Z = \mathrm{Tr}\left[\hat{U}(0,\beta)\right]$
- Generator of connected functions:
  $W[j_i^*(\tau), j_i(\tau)] = \log Z = -\beta F[j_i^*(\tau), j_i(\tau)]$

Generating Functionals

- Symmetry breaking source:

$$\hat{H}_{\mathrm{BH}}(\tau) = \hat{H}_{\mathrm{BH}} + \sum_i \left[j_i^*(\tau)\hat{a}_i + j_i(\tau)\hat{a}_i^\dagger \right]$$

- Evolution operator: $\hat{U}(\tau,\tau_0) = \hat{T} e^{-\int_{\tau_0}^{\tau} d\tau' \hat{H}_{\mathrm{BH}}(\tau)}$
- Generalized partition function: $Z = \mathrm{Tr}\left[\hat{U}(0,\beta)\right]$
- Generator of connected functions:
  $W[j_i^*(\tau), j_i(\tau)] = \log Z = -\beta F[j_i^*(\tau), j_i(\tau)]$
- Order parameter field: $\psi_i(\tau) = \beta \frac{\delta F}{\delta j_i^*(\tau)}$

Generating Functionals

- Symmetry breaking source:

$$\hat{H}_{\mathrm{BH}}(\tau) = \hat{H}_{\mathrm{BH}} + \sum_i \left[j_i^*(\tau)\hat{a}_i + j_i(\tau)\hat{a}_i^\dagger \right]$$

- Evolution operator: $\hat{U}(\tau,\tau_0) = \hat{T} e^{-\int_{\tau_0}^{\tau} d\tau' \hat{H}_{\mathrm{BH}}(\tau)}$
- Generalized partition function: $Z = \mathrm{Tr}\left[\hat{U}(0,\beta)\right]$
- Generator of connected functions:
  $W[j_i^*(\tau), j_i(\tau)] = \log Z = -\beta F[j_i^*(\tau), j_i(\tau)]$
- Order parameter field: $\psi_i(\tau) = \beta \frac{\delta F}{\delta j_i^*(\tau)}$
- Effective action:
  $\Gamma[\psi_i^*(\tau), \psi_i(\tau)] = F - \frac{1}{\beta}\sum_i \int_0^\infty d\tau \left[\psi_i^*(\tau) j_i(\tau) + \psi_i(\tau) j_i^*(\tau)\right]$

Generating Functionals

- Symmetry breaking source:

$$\hat{H}_{\mathrm{BH}}(\tau) = \hat{H}_{\mathrm{BH}} + \sum_i \left[j_i^*(\tau)\hat{a}_i + j_i(\tau)\hat{a}_i^\dagger \right]$$

- Evolution operator: $\hat{U}(\tau,\tau_0) = \hat{T} e^{-\int_{\tau_0}^{\tau} d\tau' \hat{H}_{\mathrm{BH}}(\tau)}$
- Generalized partition function: $Z = \mathrm{Tr}\left[\hat{U}(0,\beta)\right]$
- Generator of connected functions:
  $W[j_i^*(\tau), j_i(\tau)] = \log Z = -\beta F[j_i^*(\tau), j_i(\tau)]$
- Order parameter field: $\psi_i(\tau) = \beta \frac{\delta F}{\delta j_i^*(\tau)}$
- Effective action:
  $\Gamma[\psi_i^*(\tau), \psi_i(\tau)] = F - \frac{1}{\beta}\sum_i \int_0^\infty d\tau \left[\psi_i^*(\tau) j_i(\tau) + \psi_i(\tau) j_i^*(\tau)\right]$
- Equation of motion: $\frac{\delta \Gamma}{\delta \psi_i(\tau)} = 0$

Freie Universität Berlin

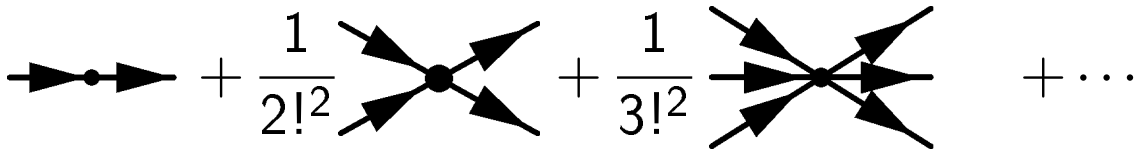
Outline

Diagrammatic Expansion

- General formula: $Z\left[j^*,j\right] = e^{\sum_{ii'} J_{ii'} \int_0^\beta d\tau \frac{\delta}{\delta j_i^*(\tau)\delta j_{i'}(\tau)}} Z_0\left[j^*,j\right]$

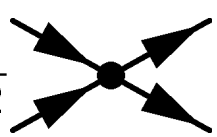
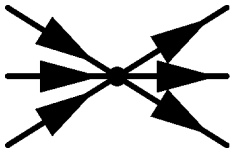
Diagrammatic Expansion

- General formula: $Z[j^*,j] = e^{\sum_{ii'} J_{ii'} \int_0^\beta d\tau \frac{\delta}{\delta j_i^*(\tau)\delta j_{i'}(\tau)}} Z_0[j^*,j]$
- Diagrammatic representation:

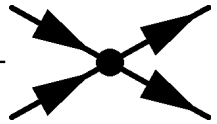
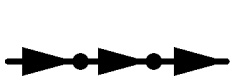
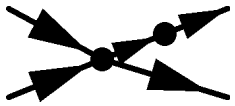
$W_0[j^*,j] =$ 

Diagrammatic Expansion

- General formula: $Z[j^*,j] = e^{\sum_{ii'} J_{ii'} \int_0^\beta d\tau \frac{\delta}{\delta j_i^*(\tau)\delta j_{i'}(\tau)}} Z_0[j^*,j]$
- Diagrammatic representation:

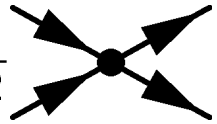
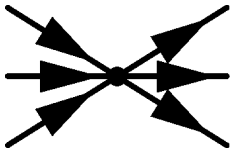
$W_0[j^*,j] = \bullet +$  $+ \frac{1}{2!^2}$  $+ \frac{1}{3!^2}$  $+ \cdots$

- Perturbative expansion:

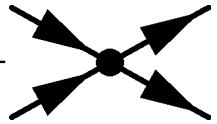
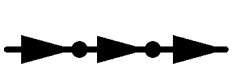
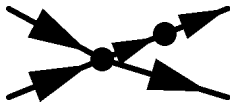
$W[j^*,j] = \bullet +$  $+ \frac{1}{2!^2}$  $+$  $+ \frac{1}{2!}$  $+ \cdots$

Diagrammatic Expansion

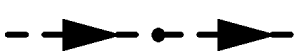
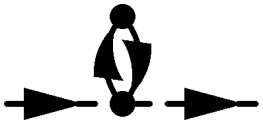
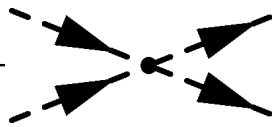
- General formula: $Z[j^*,j] = e^{\sum_{ii'} J_{ii'} \int_0^\beta d\tau \frac{\delta}{\delta j_i^*(\tau)\delta j_{i'}(\tau)}} Z_0[j^*,j]$
- Diagrammatic representation:

$W_0[j^*,j] = \bullet +$  $+ \frac{1}{2!^2}$  $+ \frac{1}{3!^2}$  $+ \cdots$

- Perturbative expansion:

$W[j^*,j] = \bullet +$  $+ \frac{1}{2!^2}$  $+$  $+ \frac{1}{2!}$  $+ \cdots$

- Effective action has only 1PI diagrams

$-\beta\Gamma[\psi^*,\psi] = \Gamma^{(0)} +$  $+$  $+ \frac{1}{2!^2}$  $+ \cdots$

Outline

Diagrammatic Rules

- Vertices represents connected functions $W^{(n)}(\tau', \tau)$

Diagrammatic Rules

- Vertices represents connected functions $W^{(n)}(\tau', \tau)$
- Each vertex means a sum over all lattice sites

Diagrammatic Rules

- Vertices represents connected functions $W^{(n)}(\tau', \tau)$
- Each vertex means a sum over all lattice sites
- Internal lines multiplies by the hopping matrix

Diagrammatic Rules

- Vertices represents connected functions $W^{(n)}(\tau', \tau)$
- Each vertex means a sum over all lattice sites
- Internal lines multiplies by the hopping matrix
- External lines inward(outward) lines multiplies $j^*(j)$

Diagrammatic Rules

- Vertices represents connected functions $W^{(n)}(\tau', \tau)$
- Each vertex means a sum over all lattice sites
- Internal lines multiplies by the hopping matrix
- External lines inward(outward) lines multiplies $j^*(j)$
- Each line means an integral from 0 to β

Diagrammatic Rules

- Vertices represents connected functions $W^{(n)}(\tau', \tau)$
- Each vertex means a sum over all lattice sites
- Internal lines multiplies by the hopping matrix
- External lines inward(outward) lines multiplies $j^*(j)$
- Each line means an integral from 0 to β
- Symmetry factor is number of ways of joining vertices and lines

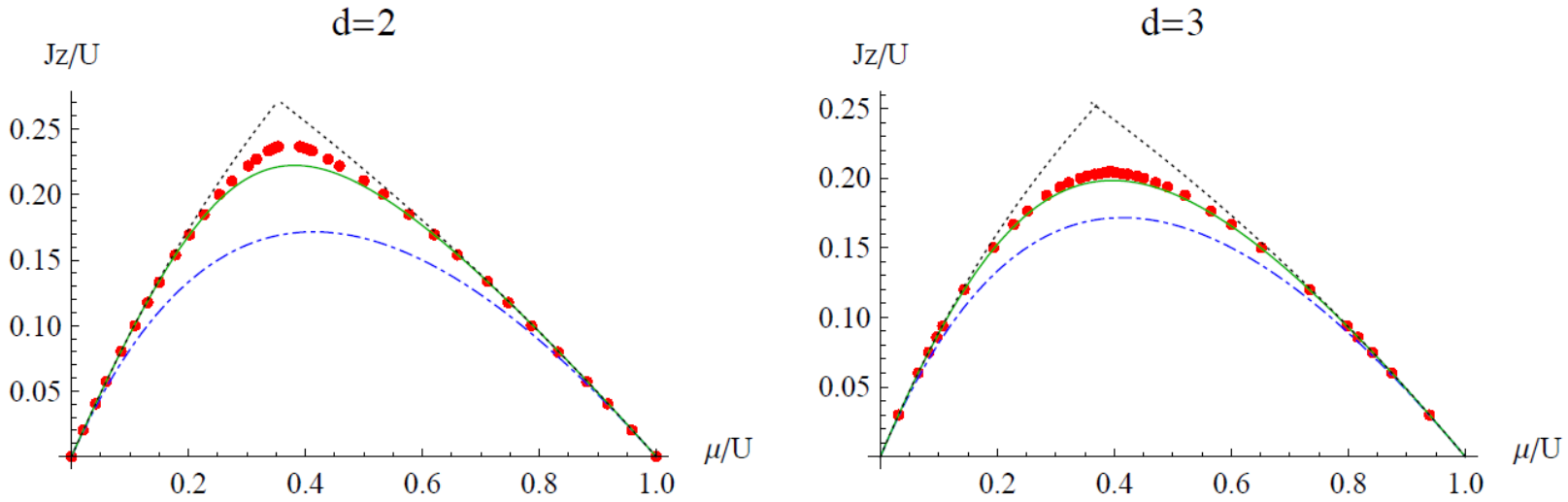
Outline

Quantum Phase Diagram

- System enters the MI phase when ψ_{eq} vanishes

Quantum Phase Diagram

- System enters the MI phase when ψ_{eq} vanishes
- Phase diagram at second hopping order

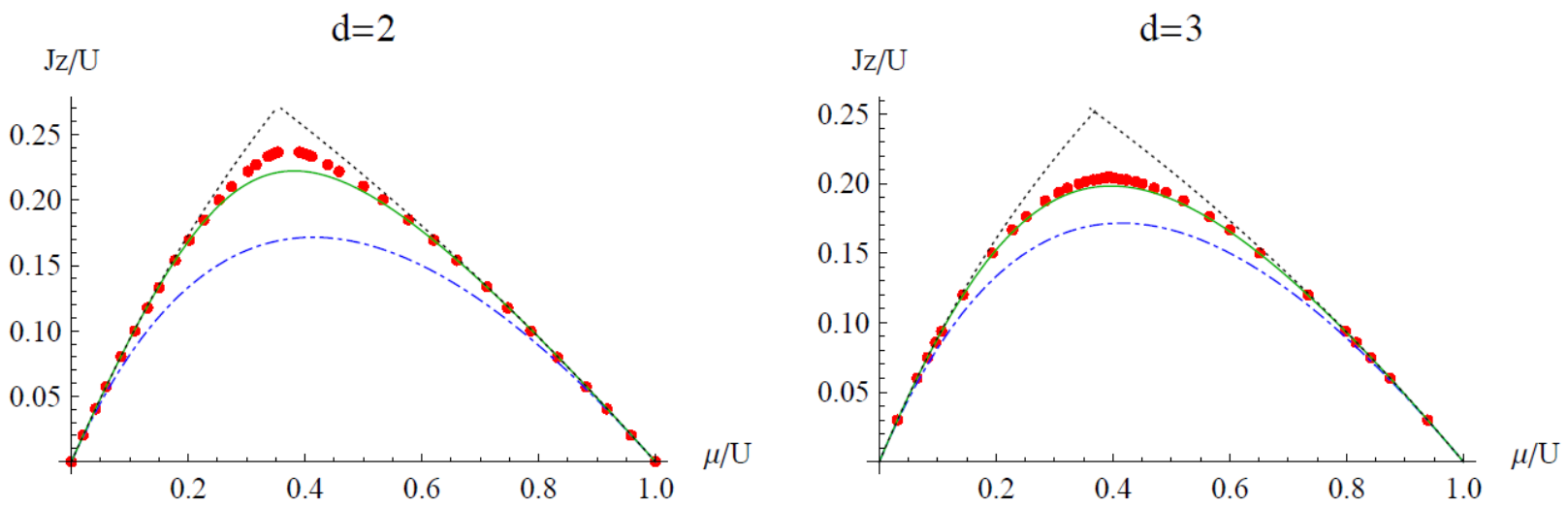


F.E.A. Santos, and A. Pelster PRA **79**:013614, 2009

Freie Universität Berlin

Quantum Phase Diagram

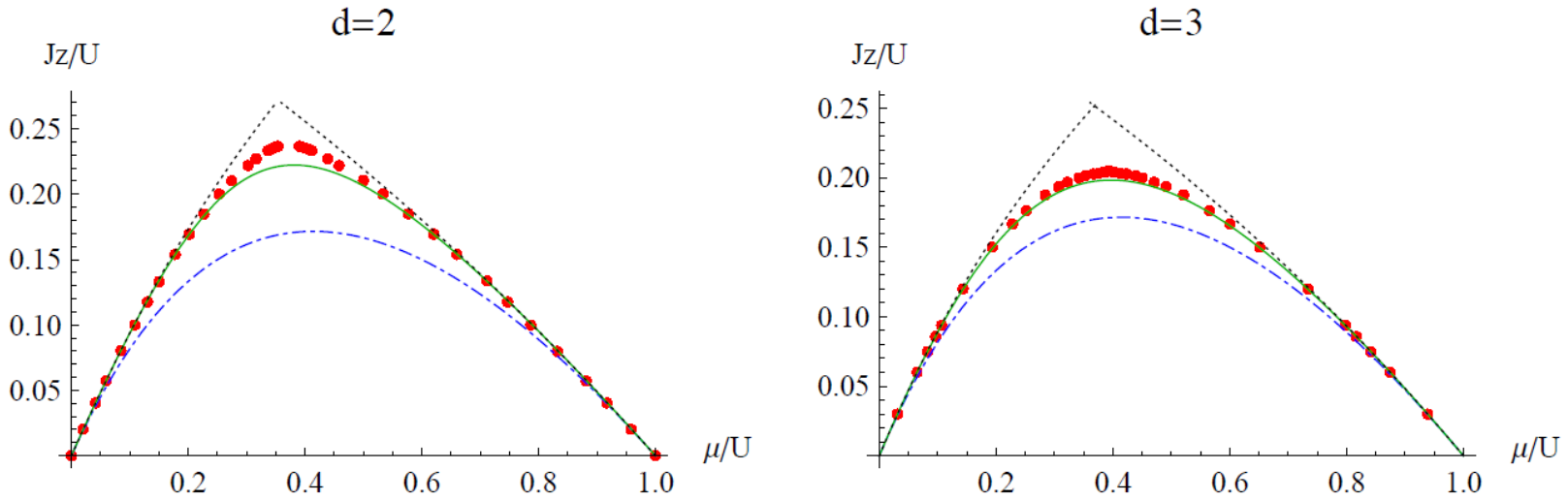
- System enters the MI phase when ψ_{eq} vanishes
- Phase diagram at second hopping order
- Error smaller than 3% in 3D



F.E.A. Santos, and A. Pelster PRA **79**:013614, 2009

Quantum Phase Diagram

- System enters the MI phase when ψ_{eq} vanishes
- Phase diagram at second hopping order
- Error smaller than 3% in 3D
- Fast convergence : N. Teichmann et. al. PRB **79**: 195131

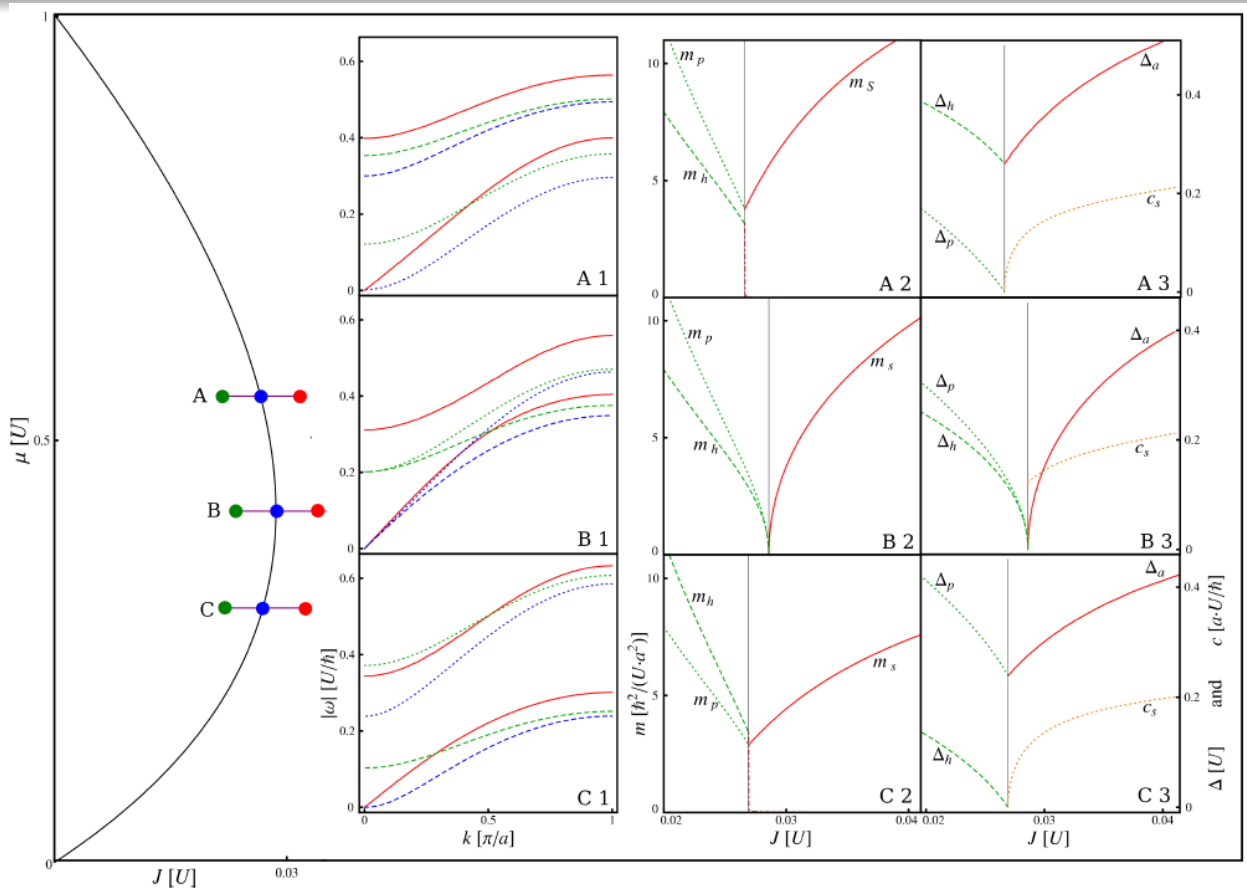


F.E.A. Santos, and A. Pelster PRA **79**:013614, 2009

Outline

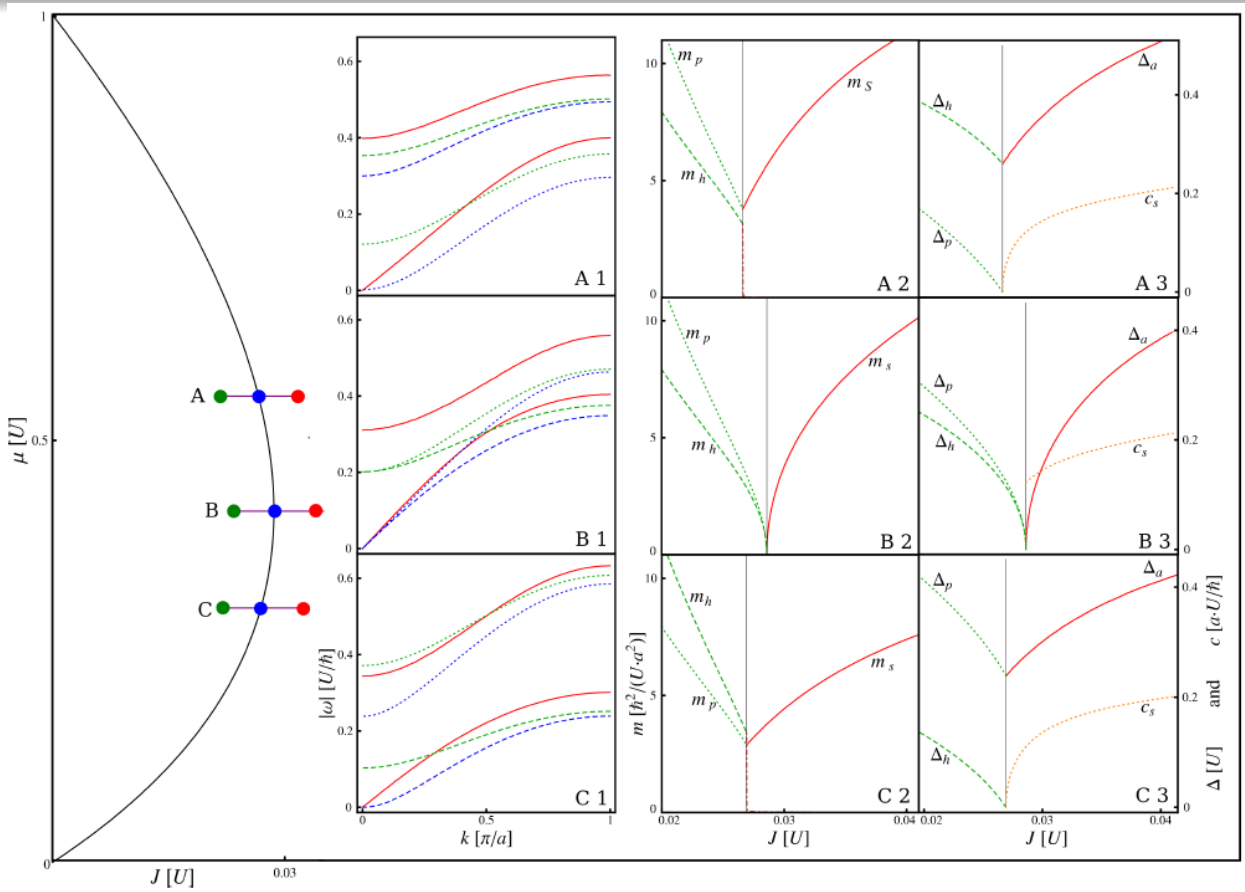
Excitation Spectra

- MI phase: Particle and hole excitations



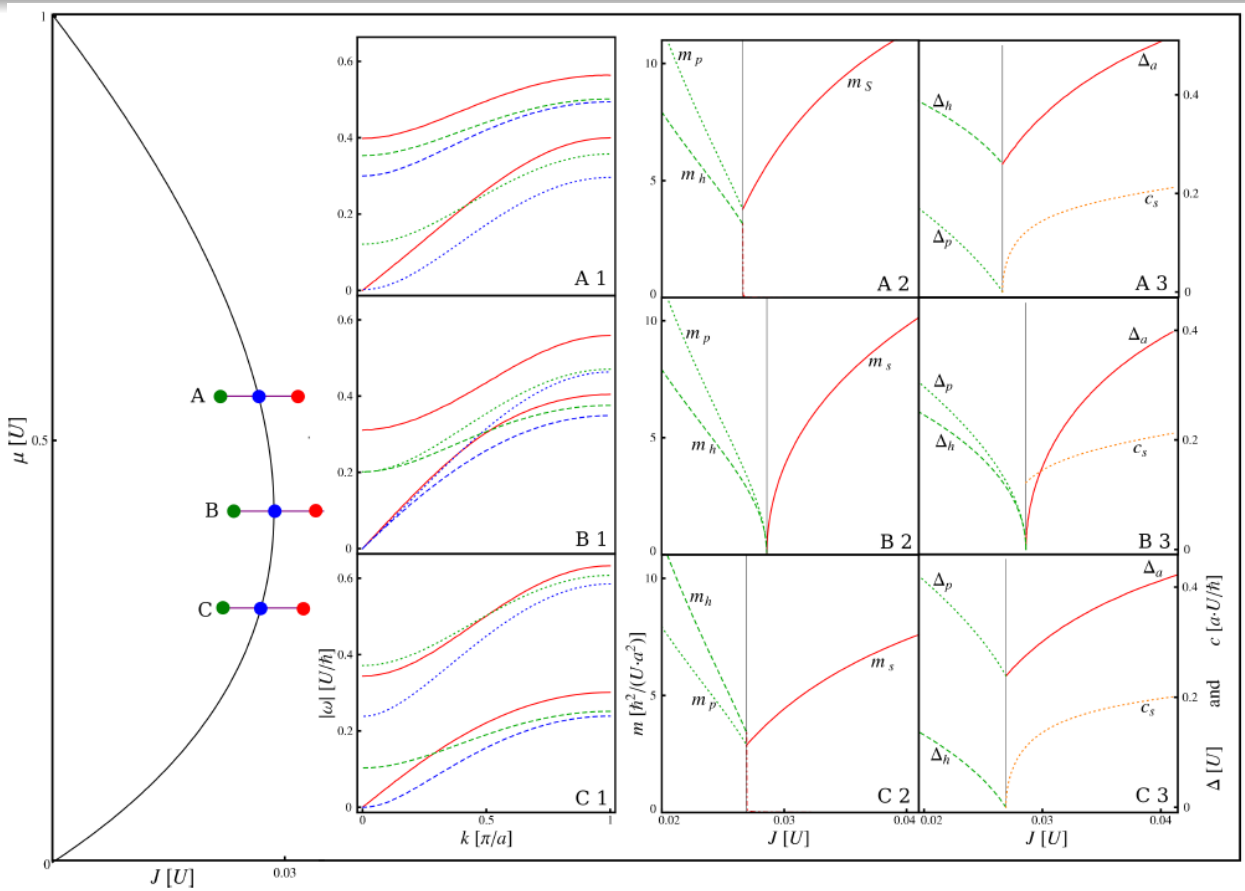
Excitation Spectra

- MI phase: Particle and hole excitations
- SF phase: Density and phase excitations



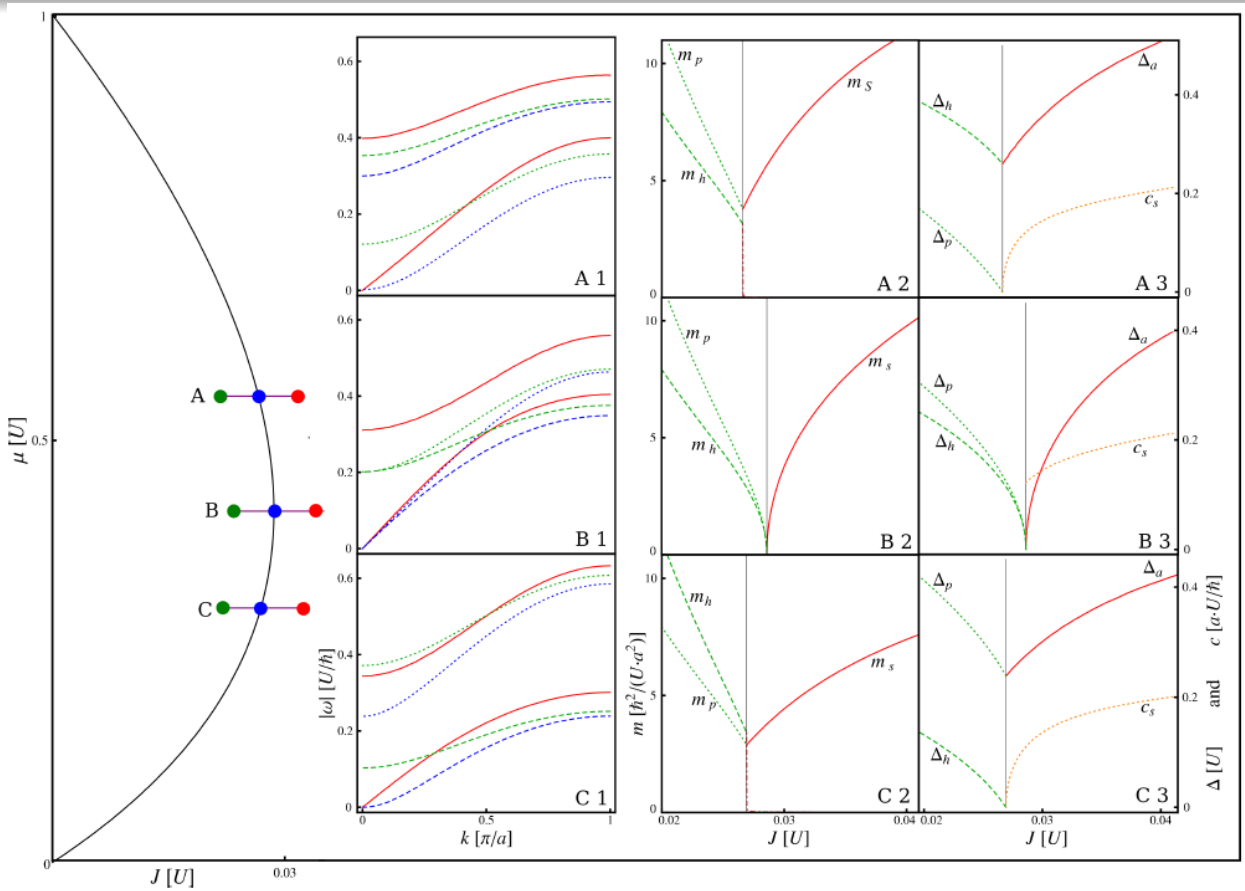
Excitation Spectra

- MI phase: Particle and hole excitations
- SF phase: Density and phase excitations
- Different universality class at the tip



Excitation Spectra

- MI phase: Particle and hole excitations
- SF phase: Density and phase excitations
- Different universality class at the tip

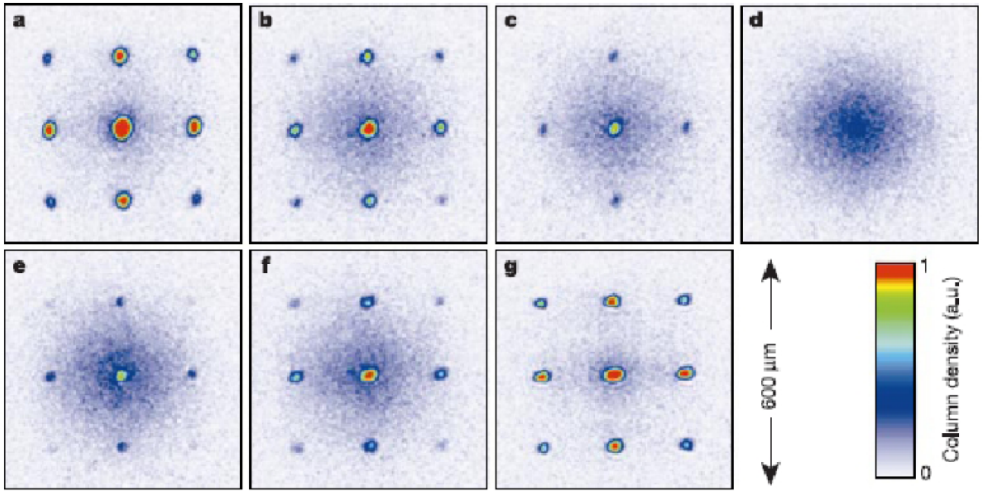


B. Bradlyn, F.E.A. Santos, and A. Pelster PRA **79**:013615, 2009
T.D. Grass, F.E.A. Santos, and A. Pelster Laser Physics **21**:1459, 2011
T.D. Grass, F.E.A. Santos, and A. Pelster PRA **84**:013613, 2011

Outline

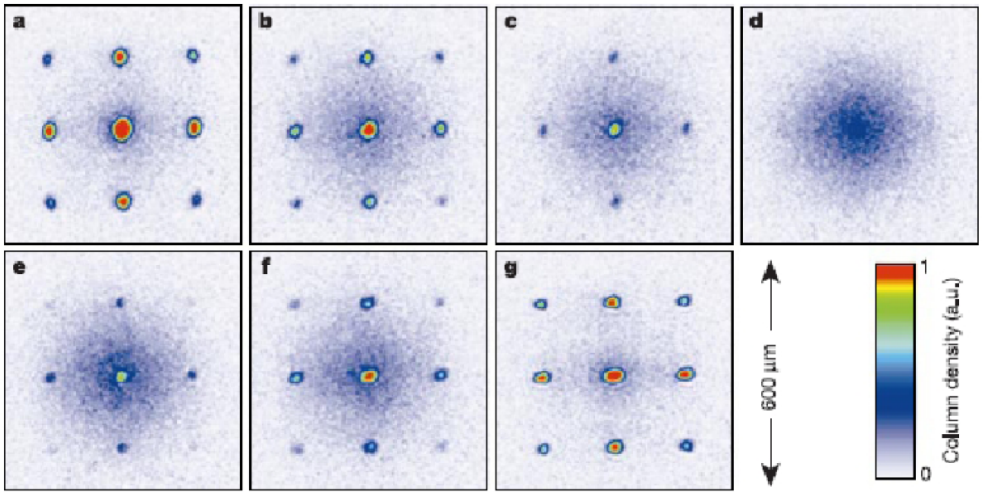
Collapse and Revival of Matter Waves

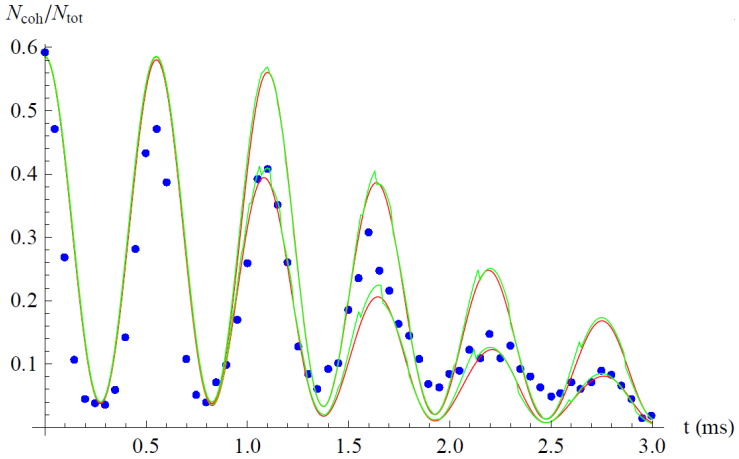
- Sample of 2×10^5 ^{87}Rb atoms:
  M. Greiner et. al. Nature 419, 51 (2002)
- Periodic potential depth suddenly changed from $8E_R$ to $22E_R$



Collapse and Revival of Matter Waves

- Sample of 2×10^5 ^{87}Rb atoms:
  M. Greiner et. al. Nature 419, 51 (2002)
- Periodic potential depth suddenly changed from $8E_R$ to $22E_R$
- Inhomogeneous chemical potential: $\mu \to \mu - \frac{m}{2}\omega^2 \mathbf{r}_i$
- Wick rotation $\tau \to it$





F.E.A. dos Santos, and A. Pelster in preparation **84**:013613, 2011

Summary and Outlook

- A Ginzburg-Landau theory was developed for bosons in optical lattices
- Analytical calculation are performed using diagrammatic methods
- High accuracy to equilibrium and out-of-equilibrium systems

- Outlook
  - Different geometries
  - Bose-Fermi mixtures
  - Optical QED lattices