

# BECs with $1/r$ Interatomic Interaction



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- 1. Experimental Proposal**
- 2. Hartree-Fock Mean-Field Theory**
- 3. Shift of Critical Temperature**

# 1. Experimental Proposal: Theoretical Basis

## System of atoms and radiation field

Interatomic contribution in 4th order QED perturbation theory:

$$\Delta E^{(4)} = \frac{I}{4\pi\varepsilon_0^2 c} \hat{e}_i^*(\mathbf{k}) \hat{e}_j(\mathbf{k}) \alpha^2(k) V_{ij}(\mathbf{r}, k) \cos(\mathbf{k} \cdot \mathbf{r})$$

With retarded dipole-dipole interaction tensor:

$$V_{ij}(\mathbf{r}, k) = \frac{1}{4\pi\varepsilon_0 r^3} [(\delta_{ij} - 3\hat{r}_i\hat{r}_j)(\cos kr + kr \sin kr) - (\delta_{ij} - \hat{r}_i\hat{r}_j)k^2 r^2 \cos kr]$$

Rotational average in near zone  $kr \ll 1$ :  $\Delta E^{(4)} \propto -\frac{1}{r}$

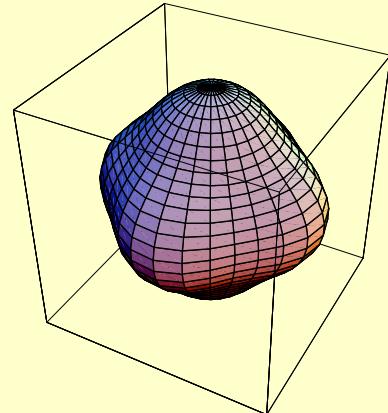
D.P. Craig and T. Thirunamachandran, *Molecular Quantum Electrodynamics* (Academic Press, London, 1984)

# Realization of Rotational Average

## Static: 3 lasers

$$\mathbf{k}_1 = k\mathbf{e}_x, \mathbf{k}_2 = k\mathbf{e}_y, \mathbf{k}_3 = k\mathbf{e}_z$$

$$U_3(r) = -\frac{3Ik^2\alpha^2}{16\pi c\varepsilon_0^2} \frac{1}{r} \left[ \frac{7}{3} + (\sin \vartheta \cos \varphi)^4 + (\sin \vartheta \sin \varphi)^4 + (\cos \vartheta)^4 \right]$$



**pure  $1/r$ : 18 lasers**

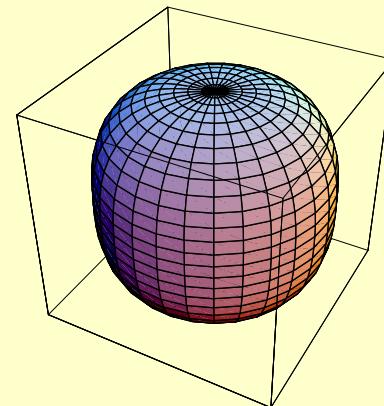
PRL 84, 5687 (2000)

## Rotating: 1 laser

$$\mathbf{k} = k(\sin \gamma \cos \Omega t, \sin \gamma \sin \Omega t, \cos \gamma)$$

$$\cos^2 \gamma = \frac{1}{3}, \quad \Omega \ll \Omega_{\text{laser}}$$

$$U_{\text{rot}}(r) = -\frac{I\alpha^2 k^2}{96\pi c\varepsilon_0^2} \frac{1}{r} (17 + 6 \cos^2 \vartheta - 7 \cos^4 \vartheta)$$



**pure  $1/r$ : 3 lasers**

**our proposal**

## 2. Hartree-Fock Mean-Field Theory

**Partition function:**

$$\mathcal{Z} = \oint \mathcal{D}\psi^* \oint \mathcal{D}\psi \exp \left\{ -\frac{1}{\hbar} \mathcal{A}_0[\psi, \psi^*] - \frac{1}{\hbar} \mathcal{A}_{\text{int}}[\psi, \psi^*] \right\}$$

**Action:**

$$\mathcal{A}_0 [\psi, \psi^*] = \int_0^{\hbar\beta} d\tau \int d^D x \psi^*(\mathbf{x}, \tau) \left[ \hbar \frac{\partial}{\partial \tau} - \frac{\hbar^2}{2M} \Delta + V^{(\text{ext})}(\mathbf{x}) - \mu \right] \psi(\mathbf{x}, \tau)$$

$$\mathcal{A}_{\text{int}} [\psi, \psi^*] = \frac{1}{2} \int_0^{\hbar\beta} d\tau \int d^D x \int d^D x' V^{(\text{int})}(\mathbf{x}, \mathbf{x}') |\psi(\mathbf{x}, \tau)|^2 |\psi^*(\mathbf{x}', \tau)|^2$$

$$V^{(\text{int})}(\mathbf{x}, \mathbf{x}') = g \delta(\mathbf{x} - \mathbf{x}') - \frac{u}{|\mathbf{x} - \mathbf{x}'|}$$

## Background method

$$\psi(\mathbf{x}, \tau) = \Psi(\mathbf{x}, \tau) + \delta\psi(\mathbf{x}, \tau) \quad , \quad \psi^*(\mathbf{x}, \tau) = \Psi^*(\mathbf{x}, \tau) + \delta\psi^*(\mathbf{x}, \tau)$$

$$\oint \mathcal{D}\psi \rightarrow \oint \mathcal{D}\delta\psi \quad , \quad \oint \mathcal{D}\psi^* \rightarrow \oint \mathcal{D}\delta\psi^*$$

**Problematic term: Quartic in fields**

**Reduced to quadratic form by Gaussian approximation**

$$\delta\psi^*(\mathbf{x}, \tau)\delta\psi^*(\mathbf{x}', \tau)\delta\psi(\mathbf{x}, \tau)\delta\psi(\mathbf{x}', \tau)$$

$$\begin{aligned} &\approx \langle \delta\psi^*(\mathbf{x}, \tau)\delta\psi(\mathbf{x}, \tau) \rangle \delta\psi^*(\mathbf{x}', \tau)\delta\psi(\mathbf{x}', \tau) + \langle \delta\psi^*(\mathbf{x}, \tau)\delta\psi(\mathbf{x}', \tau) \rangle \delta\psi^*(\mathbf{x}', \tau)\delta\psi(\mathbf{x}, \tau) \\ &+ \langle \delta\psi^*(\mathbf{x}', \tau)\delta\psi(\mathbf{x}, \tau) \rangle \delta\psi^*(\mathbf{x}, \tau)\delta\psi(\mathbf{x}', \tau) + \langle \delta\psi^*(\mathbf{x}', \tau)\delta\psi(\mathbf{x}', \tau) \rangle \delta\psi^*(\mathbf{x}, \tau)\delta\psi(\mathbf{x}, \tau) \\ &- \langle \delta\psi^*(\mathbf{x}', \tau)\delta\psi(\mathbf{x}, \tau) \rangle \langle \delta\psi^*(\mathbf{x}, \tau)\delta\psi(\mathbf{x}', \tau) \rangle - \langle \delta\psi^*(\mathbf{x}', \tau)\delta\psi(\mathbf{x}', \tau) \rangle \langle \delta\psi^*(\mathbf{x}, \tau)\delta\psi(\mathbf{x}, \tau) \rangle \end{aligned}$$

$$\implies \mathcal{A} \approx \mathcal{A}_{\text{BG}} [\Psi, \Psi^*] + \mathcal{A}^{(2)} [\delta\psi^*, \delta\psi, \Psi, \Psi^*]$$

**Effective action:**  $\Gamma[\Psi, \Psi^*, g] = -\frac{1}{\beta} \ln \mathcal{Z}$ ,  $g(\mathbf{x}, \tau; \mathbf{x}', \tau') = \langle \delta\psi(\mathbf{x}, \tau)\delta\psi^*(\mathbf{x}', \tau') \rangle$

**Free energy  $\mathcal{F}$  as minimum of  $\Gamma[\Psi, \Psi^*, g]$ :**

$$\frac{\delta\Gamma[\Psi, \Psi^*, g]}{\delta\Psi^*(\mathbf{x}, \tau)} = 0, \quad \frac{\delta\Gamma[\Psi, \Psi^*, g]}{\delta g(\mathbf{x}, \tau; \mathbf{x}', \tau')} = 0$$

$$(I) : \quad 0 = \left\{ \hbar \frac{\partial}{\partial \tau} - \frac{\hbar^2}{2M} \Delta + V^{(\text{ext})}(\mathbf{x}) - \mu + \int d^D x' |\Psi(\mathbf{x}', \tau)|^2 V^{(\text{int})}(\mathbf{x}, \mathbf{x}') \right\} \Psi(\mathbf{x}, \tau)$$

$$+ \int d^D x' V^{(\text{int})}(\mathbf{x}, \mathbf{x}') \left[ \Psi(\mathbf{x}, \tau) g(\mathbf{x}', \tau; \mathbf{x}', \tau) + \Psi(\mathbf{x}', \tau) g(\mathbf{x}', \tau; \mathbf{x}, \tau) \right]$$

$$(II) : \quad g(\mathbf{x}, \tau; \mathbf{x}', \tau') = \langle \delta\psi(\mathbf{x}, \tau) \delta\psi^*(\mathbf{x}', \tau') \rangle$$

$T \rightarrow 0$  leads to **Gross-Pitaevskii equation:** PRL 84, 5687 (2000)

### 3. Shift of Critical Temperature

**Calculation with two methods:**

- Thermodynamic limit from Hartree-Fock mean-field theory
- Feynman many-body perturbation theory

**For fixed particle number**

$$N = -\frac{\partial \mathcal{F}}{\partial \mu}$$

**and cylinder-symmetric trap**

$$V^{(\text{ext})}(\mathbf{x}) = \frac{m}{2} [\omega_r^2 r^2 + \omega_z^2 z^2]$$

**Procedure:**  $N = N(\mu), \quad \mu \uparrow \mu_c \quad \Rightarrow \quad T_c$

# Feynman perturbation theory

**Feynman rules:**  $\mathbf{x}, \tau \longleftrightarrow \mathbf{x}', \tau' \equiv G^{(0)}(\mathbf{x}, \tau; \mathbf{x}', \tau')$

$$\text{Diagram: Two external lines labeled } \mathbf{x}, \tau \text{ and } \mathbf{x}', \tau' \text{ meeting at a vertex labeled } \tau. \quad \equiv -\frac{1}{\hbar} \int_0^{\hbar\beta} d\tau \int d^3x \int d^3x' V^{(\text{int})}(\mathbf{x} - \mathbf{x}')$$

**Free energy:**  $\mathcal{F} = \mathcal{F}^{(0)} - \frac{1}{\beta} \left\{ \frac{1}{2} \text{Diagram: Two circles connected by a dashed line} + \frac{1}{2} \text{Diagram: A single circle with a self-energy loop} + \dots \right\}$

**Self energy:**  $\Sigma(\mathbf{x}, \tau; \mathbf{x}', \tau') = \text{Diagram: A circle with a vertical dashed line connecting it to two external lines labeled } \mathbf{x}, \tau \text{ and } \mathbf{x}', \tau' + \text{Diagram: Two external lines labeled } \mathbf{x}, \tau \text{ and } \mathbf{x}', \tau' \text{ with a curved self-energy loop between them} + \dots$

$$\mu_c = -\hbar \Sigma(\mathbf{p} = \mathbf{0}, \omega_m = 0, (\mathbf{x} + \mathbf{x}')/2 = \mathbf{0})$$

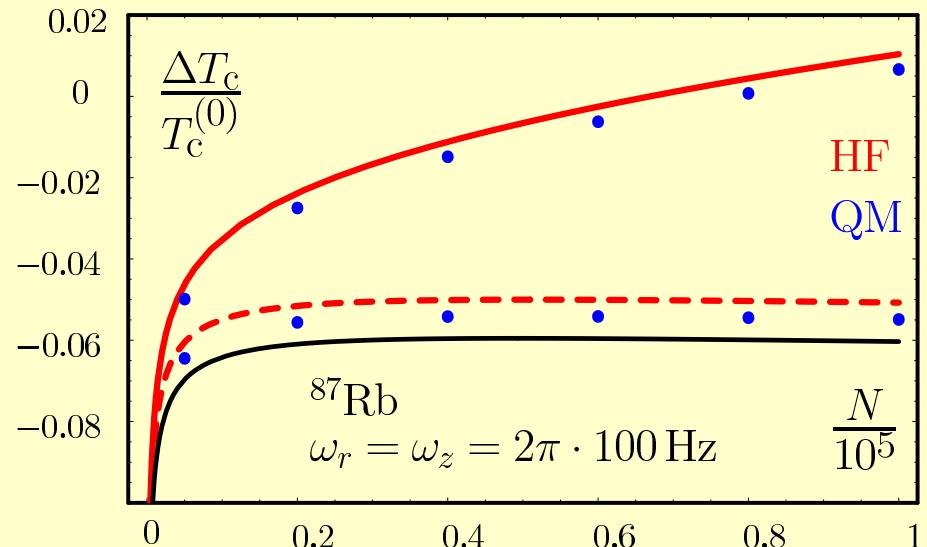
**Note: Semiclassics not applicable**

⇒ **using full quantum-mechanical correlation function**

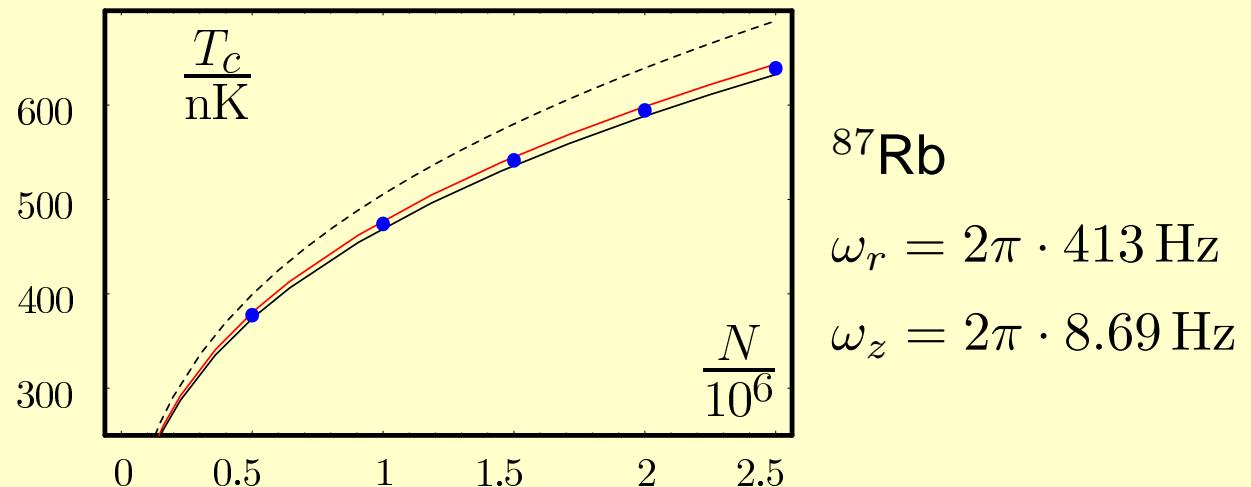
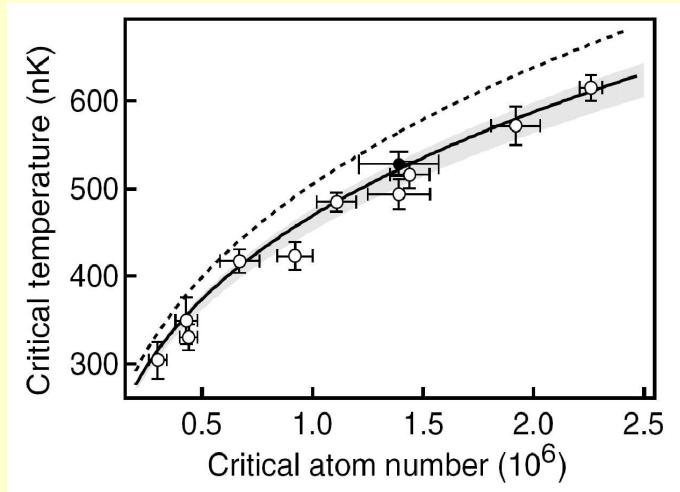
## Results of critical temperature shift:

$$\frac{\Delta T_c}{T_c^{(0)}} = -c_\delta(N) \frac{a}{\lambda_c} + \frac{\lambda_c}{a_G} \left[ c_E(N) + c_D \frac{1}{(\hbar \beta_c^{(0)} \omega)^2} \right]$$

$$\lambda_c = \sqrt{\frac{2\pi\hbar^2\beta_c^{(0)}}{M}}, \quad a_G = \frac{4\pi^2\hbar^2}{uM}$$



## Vanishing $1/r$ interaction: Comparison with experiment



PRL 92, 030405 (2004)