Vortex Lines in an Imperfect Bose Gas

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It is shown that the vortex lines possessing a thickness which is inversely proportional to the square root of the gas density and of the intensity of the interaction may exist in Bose gases with weak repulsion between the atoms. The energy of a vortex line is computed. It is also shown that in the presence of a vortex line a branch appears in the energy spectrum of the gas which corresponds to oscillations of the vortex.

Overview

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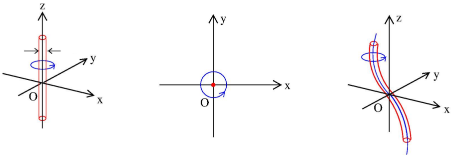
Introduction

Onsager-Feynman Vortex

Onsager-Feynman Vortex

The Feynman [2]–Onsager [3] proposed quantized vortex. Experimental observation realized in 1956 by Hall and Vinen [4].

$$\mathbf{v} = s\frac{\hbar}{mr}\hat{\theta} \tag{1}$$



$$\Gamma = \oint_{\lambda} \mathbf{v}\cdot \mathrm{d}\boldsymbol{\lambda} = s\frac{h}{m}, \quad s = \pm 1, 2, 3\ldots \tag{2}$$

C. Barenghi and N. G. Parker, *A primer on quantum fluids,* (Springer International Publishing, 2016).

Introduction

Energy Contribution of the Vortex

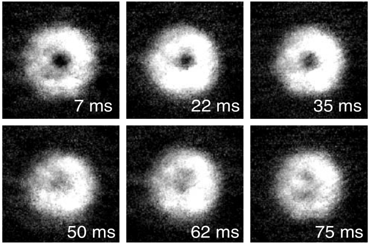
Energy Contribution of the Vortex

The energy per unit length contribution of a vortex can be estimated by considering a cylinder of length L and radius R with the vortex at the center:

$$E = \frac{m}{2} \int_{r_0}^{R} n(\mathbf{v} \cdot \mathbf{v}) r \, \mathrm{d}r \, \mathrm{d}\theta = s^2 \frac{\pi \hbar^2 n}{m} \log \frac{R}{r_0} \tag{3}$$

That indicates that the energy contribution of a vortex with strength $s > 1$ is always larger then the contribution of s vortices of strength 1 since:

$$s^2 - s > 0 \Rightarrow s(s-1) > 0 = \begin{cases} s > 1, & \text{if } s > 0 \\ s < -1, & \text{if } s < 0 \end{cases} \tag{4}$$



Y. Shin et al., **"Dynamical instability of a doubly quantized vortex in a bose-einstein condensate,"** en, Physical Review Letters 93, `10.1103/physrevlett.93.160406` (2004).

Introduction

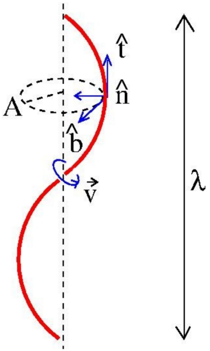
Vibration of a vortex line

Frequency of vortex line vibrations

A vortex line can vibrate, the classical problem was originally investigated by Thomson (Lord Kelvin) in the 1890s [7]. The quantized vortex matches quite well with the classical case of an “Irrotational Vortex” then, when the wavelength $\lambda \gg r_0$, the dispersion relationship follows the same law obtained by Thomson:

$$\omega(k) = \frac{\hbar k^2}{2m} \log \frac{1}{k r_0} \tag{5}$$

$$k = \frac{2\pi}{\lambda} \tag{6}$$



The Vortex Line in an Imperfect Bose Gas

Weakly Interacting Bose Gas

The second quantized Hamiltonian for a weakly interacting Bose gas was explored in detail by Bogoliubov [8] and can be written as:

$$\hat{H} = \int \mathrm{d}^3 r \left\{ -\frac{\hbar^2}{2m} \hat{\psi}^\dagger(\mathbf{r}) \nabla^2 \hat{\psi}(\mathbf{r}) + \frac{1}{2} \int \mathrm{d}^3 r' \, \hat{\psi}^\dagger(\mathbf{r}) \hat{\psi}^\dagger(\mathbf{r'}) U(|\mathbf{r} - \mathbf{r'}|) \hat{\psi}(\mathbf{r}) \hat{\psi}(\mathbf{r'}) \right\} \tag{7}$$

The field operator $\hat{\psi}$ satisfy the commutation relations:

$$\left[\hat{\psi}(\mathbf{r}), \hat{\psi}^\dagger(\mathbf{r'})\right] = \delta(\mathbf{r} - \mathbf{r'}) \tag{8}$$

$$\left[\hat{\psi}(\mathbf{r}), \hat{\psi}(\mathbf{r'})\right] = 0 \tag{9}$$

Furthermore for a dilute gas we can approximate the inter atomic potential to a contact interaction of strength g:

$$U(|\mathbf{r} - \mathbf{r'}|) \approx g\delta(\mathbf{r} - \mathbf{r'}) \tag{10}$$

Equation of Motion

The equation of motion for the field operator $\hat{\psi}(r)$ is given by the Heisenberg equation:

$$i\hbar\frac{\partial\hat{\psi}}{\partial t} = [\hat{\psi}, \hat{H}] = [-\frac{\hbar^2}{2m}\nabla^2 + g\hat{\psi}\hat{\psi}^\dagger]\hat{\psi} \tag{11}$$

We follow with Bogoliubov approach and separate the mean field from the quantum fluctuations:

$$\hat{\psi} = \psi\hat{I} + \delta\hat{\psi} \tag{12}$$

By introducing such a substitution and keeping only terms of zeroth order in the fluctuation we obtain:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + g|\psi|^2\psi \tag{13}$$

This is the now called Gross–Pitaevskii equation (GPE). The same equation was derived independently by Gross [9] in the same year.

Time independent equation and the homogeneous solution

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + g|\psi|^2 \psi$$

We assume that the GPE allows for a steady state solution of the form:

$$\psi(t, \mathbf{r}) = e^{-iE_0 t/\hbar} \psi(\mathbf{r}) \tag{14}$$

This leads immediately to a time independent equation for $\psi(\mathbf{r})$:

$$-\frac{\hbar^2}{2m} \nabla^2 \psi - E_0 \psi + g|\psi|^2 \psi = 0 \tag{15}$$

$$-\frac{\hbar^2}{2m}\nabla^2\psi - E_0\psi + g|\psi|^2\psi = 0$$

We now look for homogeneous solutions of the time independent GPE. For that we introduce a normalization for the macroscopic wave function:

$$N = \int |\psi|^2 \, \mathrm{d}^3 r \tag{16}$$

with N being the total number of particles. Then for a homogeneous system of volume V we have a constant density of particles given by:

$$n = \frac{N}{V} \tag{17}$$

Taking a trial solution of the form $\psi = \sqrt{n}$ reduces the GPE to:

$$-E_0\sqrt{n} + gn\sqrt{n} = 0 \Rightarrow E_0 = gn \tag{18}$$

The Vortex Line in an Imperfect Bose Gas

Vortex Solution

Vortex Solution

In the presence of a vortex line located at the origin we look for solutions that are cylindrically symmetric and additionally vanishes as $r \to 0$ and are constant far away from the vortex line.

$$\psi = \sqrt{n} f(r) e^{i\varphi} \tag{19}$$

leading to

$$-\frac{\hbar^2}{2m}\nabla^2\psi - E_0\psi + g|\psi|^2\psi = 0 \Rightarrow -\frac{\hbar^2}{2mgn}\left(\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} - \frac{1}{r^2}f\right) - f + f^3 = 0 \tag{20}$$

The function $f(r)$ goes to 0 at the center of the vortex line and goes to 1 as we get far from the center, the distance at which this happens gives us the size of the vortex core:

$$r = r_0 r' \Rightarrow -\frac{\hbar^2}{2mgnr_0^2}\left(\frac{\partial^2 f}{\partial r'^2} + \frac{1}{r'}\frac{\partial f}{\partial r'} - \frac{1}{r'^2}f\right) - f + f^3 = 0 \tag{21}$$

which leads to

$$-\frac{\hbar^2}{2mgnr_0^2}\left(\frac{\partial^2 f}{\partial r'^2} + \frac{1}{r'}\frac{\partial f}{\partial r'} - \frac{1}{r'^2}f\right) - f + f^3 = 0 \tag{22}$$

by setting

$$r_0 = \frac{\hbar}{\sqrt{2mgn}} \tag{23}$$

which defines the length scale at which deformations in the density become relevant.

The equation reads now:

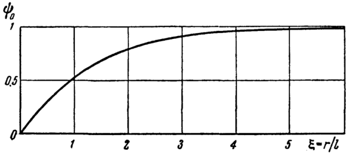
$$-\left(\frac{\partial^2 f}{\partial r'^2} + \frac{1}{r'}\frac{\partial f}{\partial r'}\right) + \left(\frac{1}{r'^2} - 1\right) f + f^3 = 0 \tag{24}$$

With the conditions:

$$\begin{aligned} f(0) &= 0 \\ f(\infty) &= 1 \end{aligned} \tag{25}$$

Pitaevskii and Ginzburg analysed this equation in a previous work where they obtained the asymptotic behaviour and the numerical solution:

$$\begin{cases} f(r') \sim r, & c_1 r \to 0 \\ f(r') \sim 1 - \frac{1}{2r'^2}, & r \to \infty \end{cases} \tag{26}$$



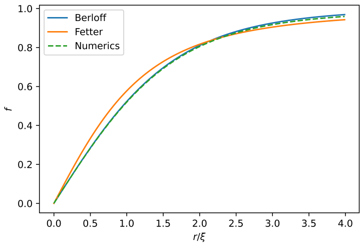
V. L. Ginzburg and L. P. Pitaevskiĭ, "On the theory of superfluidity," Soviet Physics JETP 34/7, 858–861 (1240–1245 Ž. Eksper. Teoret. Fiz.) (1958).

An interesting topic is how to construct approximate solutions for the density profile of the vortex. The first such solution was proposed by Alexander Fetter [11] and it has a very simple form:

$$f(r) = \frac{r}{\sqrt{r^2 + \alpha}} \tag{27}$$

The value of α has to be determined, Fetter did this in a variational manner and got $\alpha = 2$. Since the asymptotic series are known on both limits one can, through Padé approximation, construct higher order solutions. One such example is given by Berloff [12]:

$$f(r) = \sqrt{\frac{r^2(a_1 + a_2 r^2)}{1 + b_1 r^2 + b_2 r^4}} \tag{28}$$



As a final remark, with the numerical solution of the vortex density profile we can calculate a more accurate value for the vortex energy by solving the integral numerically

$$E = \frac{m}{2} n \int_0^{R/r_0} f(r)(\mathsf{v} \cdot \mathsf{v}) r \, \mathrm{d}r \, \mathrm{d}\theta = \frac{\pi \hbar^2 n}{m} \log 1.46 \frac{R}{r_0} \tag{29}$$

Vibrations of a condensate

Excitations in a homogeneous condensate

We now consider the following approximation

$$\hat{\psi} = (\psi_0 + \delta\psi)\hat{I} \tag{30}$$

and keep terms up to first order in the fluctuation $\delta\psi$. We also assume that ψ_0 is some constant function and all space dependencies are concentrated on the fluctuations.

$$i\hbar\frac{\partial\delta\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\delta\psi + 2g|\psi_0|^2\delta\psi + g\psi_0^2\delta\psi^* \tag{31}$$

$$-i\hbar\frac{\partial\delta\psi^*}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\delta\psi^* + 2g|\psi_0|^2\delta\psi^* + g\psi_0^{*2}\delta\psi \tag{32}$$

This was explored originally by Bogoliubov and can be solved by taking

$$\delta\psi = e^{-ignt/\hbar}\left[u_k e^{i\left(\mathbf{k}\cdot\mathbf{r}-\omega t\right)} - v_k^* e^{i\left(\mathbf{k}\cdot\mathbf{r}-\omega t\right)}\right] \tag{33}$$

Vibrations in a homogeneous condensate

This transforms the system of differential equations into a linear system of algebraic equations

$$\delta\psi = e^{-ignt/\hbar}\left[u_k e^{i\left(\mathbf{k}\cdot\mathbf{r}-\omega t\right)} - v_k^* e^{i\left(\mathbf{k}\cdot\mathbf{r}-\omega t\right)}\right] \tag{34}$$

$$\left(\frac{\hbar^2 k^2}{2m} + gn\right)u_k - gnv_k = 0 \tag{35}$$

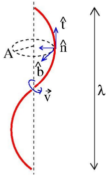
$$\left(\frac{\hbar^2 k^2}{2m} + gn + \hbar\omega\right)v_k - gnu_k = 0 \tag{36}$$

solving this system gives the well known Boguliobov dispersion relationship:

$$\hbar\omega = \pm\sqrt{\left(\frac{\hbar^2 k^2}{2m}\right)^2 + 2gn\frac{\hbar^2 k^2}{2m}} \tag{37}$$

Vibrations of a vortex line

As a final remark let us remember the vibration of a vortex line:



We have now a much more complicated system since our density is no longer constant:

$$n(\mathbf{r}) = \sqrt{n_0} f(r) \tag{38}$$

Also the proposed solution has to include both an azimutal and a longitudinal mode:

$$\delta\psi = e^{i(\varphi - gnt/\hbar)} \sum_l \left[u_l e^{i\left(kz + l\varphi - \omega t\right)} \right], l = \pm(0, 1, 2, \dots) \tag{39}$$

Which leads to a much more complicated system which has been analysed in detail by Fetter [13] and Rowland [14], but at long wavelength produces:

$$\hbar\omega = \frac{\hbar^2 k^2}{2m} \log \frac{\sqrt{2mgn_0}}{k\hbar} = \frac{\hbar^2 k^2}{2m} \log \frac{1}{kr_0} \tag{40}$$

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