Variational Perturbation Theory

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1.1. Basic Principles

Perturbation Theory:

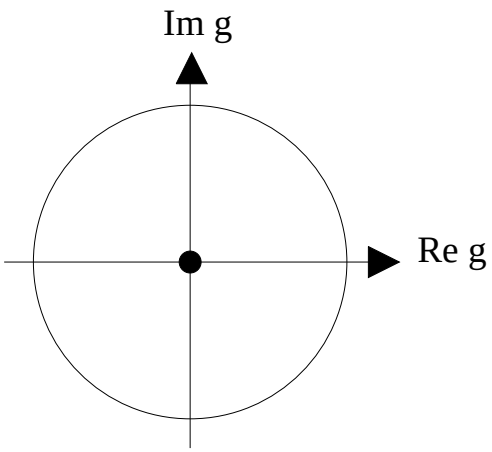
weak-coupling series: $$f_N(g) = \sum_{n=0}^{N} a_n \; g^n$$

Example:

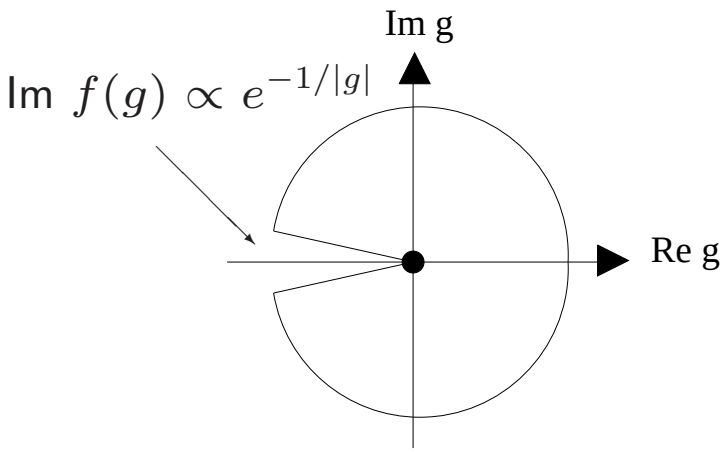
anomalous magnetic moment of electron (QED)

perturbation series: Sommerfeld fine structure constant α

Analytic Properties:



convergent series



asymptotic series

$$a_n \propto (-1)^n \, n!$$

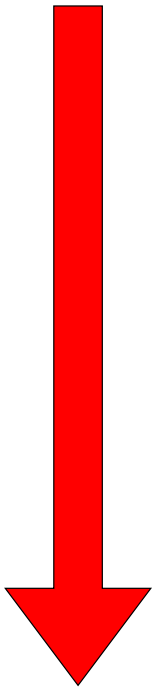
Variational Perturbation Theory:

$\Longrightarrow$ optimization with respect to variational parameters

$\Longrightarrow$ resummation of weak-coupling series

Basic Principles:

weak-coupling series: $$f_N(g) = \sum_{n=0}^{N} a_n \; g^n$$



variational perturbation theory

in the strong-coupling limit

strong-coupling series: $$f_M(g) = g^{p/q} \sum_{m=0}^{M} b_m \; g^{-2m/q}$$

Example:

	p	q	p/q	$2/q$
anharmonic oscillator	1	3	1/3	2/3

1.2. Quantum Mechanical Example

Anharmonic Oscillator:

$$V(x) = \frac{1}{2}\,\omega^2 x^2 + g\,x^4$$

Weak-Coupling Series of Ground-State Energy:

$$E_0 = \frac{\omega}{2} + g\,\frac{3}{4\omega^2} - g^2\,\frac{21}{8\omega^5} + g^3\,\frac{333}{16\omega^8} + \ldots$$

Identity:

$$V(x) = \underbrace{\frac{1}{2}\,\Omega^2 x^2}_{\text{effective harmonic oscillator}} + \underbrace{g\,x^4 + \frac{1}{2}\left(\omega^2 - \Omega^2\right) x^2}_{\text{perturbation}}$$

Ω: variational parameter

Substitution:

$$V(x) \;=\; \frac{1}{2}\left(\,\Omega\,\sqrt{1 + g\,\frac{\omega^2 - \Omega^2}{g\Omega^2}}\,\right)^2 x^2 + g\,x^4$$

$$\omega \;\Rightarrow\; \Omega\,\sqrt{1+gr}\,, \qquad r = \frac{\omega^2 - \Omega^2}{g\Omega^2}$$

First Order:

$$E_0^{(1)} = \frac{\omega}{2} + g\,\frac{3}{4\omega^2}$$

$$E_0^{(1)}(\Omega) = \frac{\Omega}{4} + \frac{\omega^2}{4\Omega} + g\,\frac{3}{4\Omega^2}$$

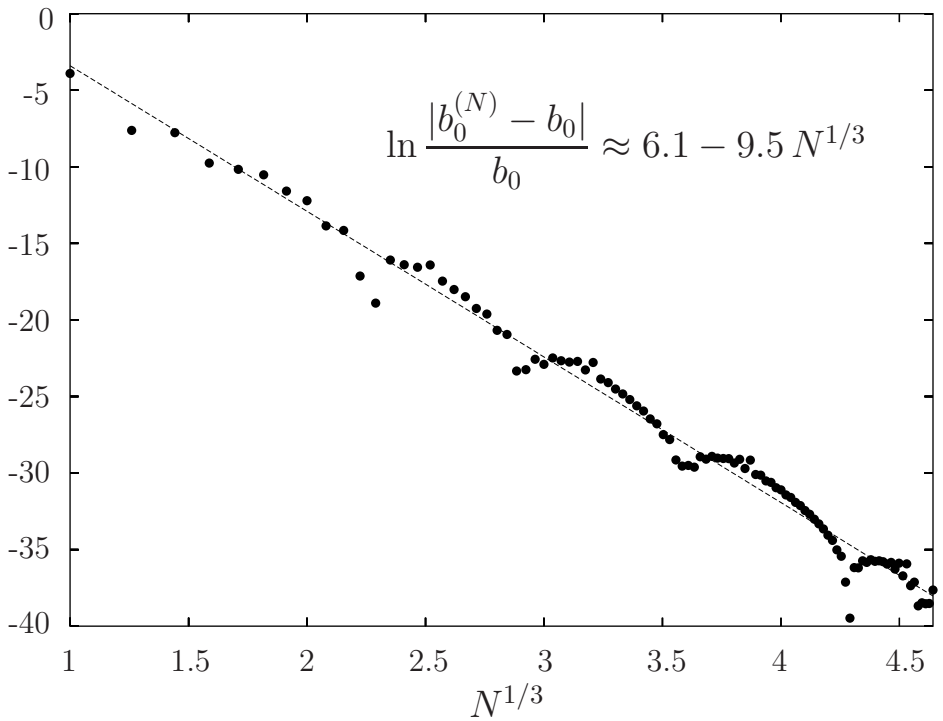
Principle of Minimal Sensitivity:

$$\frac{\partial E_0^{(1)}(\Omega)}{\partial \Omega} = 0 \quad \Rightarrow \quad \Omega^{(1)} \quad \Rightarrow \quad E_0^{(1)} \approx E_0^{(1)}(\Omega^{(1)})$$

Strong-Coupling Limit $g \to \infty$:

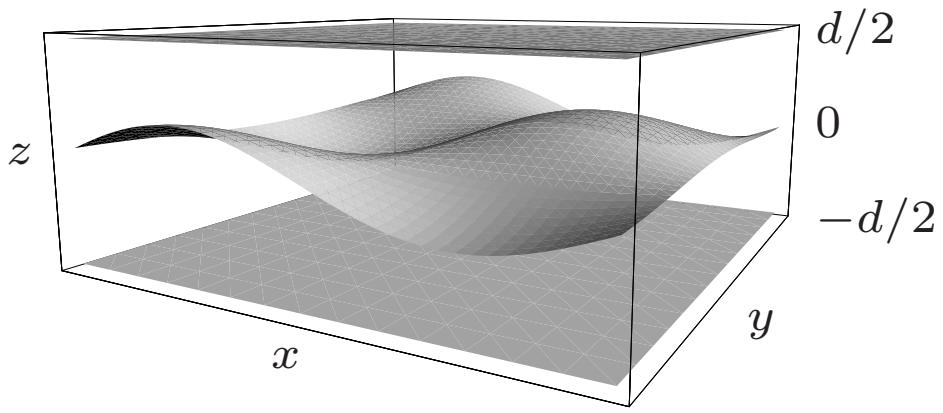
$$E_0^{(1)} = g^{1/3}\left(b_0^{(1)} + b_1^{(1)}\,g^{-2/3} + b_2^{(1)}\,g^{-4/3} + \ldots\right)$$

Exponential Convergence:



2.1. Soft Matter

Model:



Harmonic Bending Energy:

$$E = \frac{\kappa}{2} \int d^2x \left[\boldsymbol{\nabla}^2 \varphi(\mathbf{x}) \right]^2$$

κ: bending stiffness
$\varphi(\mathbf{x})$: vertical displacement at $\mathbf{x} = (x, y)$

Pressure Law:

$$p = \alpha \frac{(k_B T)^2}{\kappa (d/2)^3}$$

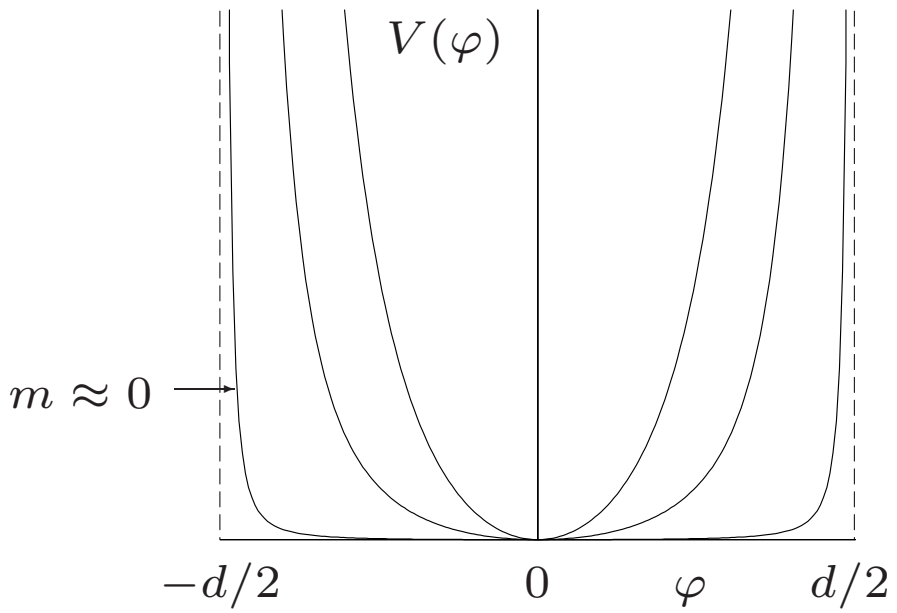
W. Helfrich: Z. Naturforschung **A 33**, 305 (1978)

Partition Function:

$$Z = \prod_{\mathbf{x}} \left[\int_{-d/2}^{+d/2} \frac{d\varphi(\mathbf{x})}{\sqrt{2\pi k_B T/\kappa}} \right] \times \exp\left\{ -\frac{\kappa}{2k_B T} \int d^2x \, [\boldsymbol{\nabla}^2 \varphi(\mathbf{x})]^2 \right\}$$

Potential Restricting Fluctuations:

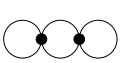
$$V(\varphi) = m^4 \frac{d^2}{\pi^2} \tan^2\left(\frac{\pi\varphi}{d}\right)$$

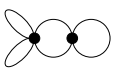
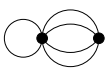


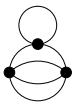
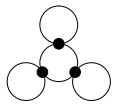
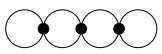
Taylor Series:

$$V(\varphi) = m^4\varphi^2 + m^4 \frac{\pi^2}{d^2} \left\{ \frac{2}{3}\varphi^4 + \frac{17}{45}\frac{\pi^2}{d^2}\varphi^6 + \ldots \right\}$$

Free Energy:

$$F^{(4)} = \frac{1}{2}$$ $$+ 3$$ $$+ 15$$ $$+ \frac{1}{2}\Big(72$$  $$+ 24$$ $$\Big)$$

$$+105$$ $$+ \frac{1}{2}\Big(540$$  $$+ 360$$  $$\Big)$$

$$+ \frac{1}{6}\Big(1728$$ $$+ 3456$$  $$+ 1728$$  $$+ 2592$$  $$\Big)$$

Weak-Coupling Series:

$$F^{(L)} = m^2 \sum_{n=0}^{L-1} a_n \left(\frac{\pi^2}{m^2 d^2}\right)^n$$

Strong-Coupling Limit ($m \to 0$): $\alpha_{\rm VPT}^{(4)} \approx 0.0797$

M. Bachmann, H. Kleinert, A. Pelster: Phys. Lett. **A 261**, 127 (1999)

Monte-Carlo Result: $\alpha_{\rm MC} = 0.0798 \pm 0.0003$

W. Janke, H. Kleinert: Phys. Lett. **A 117**, 353 (1986)
G. Gompper, D.M. Kroll: Europhys. Lett. **9**, 59 (1989)

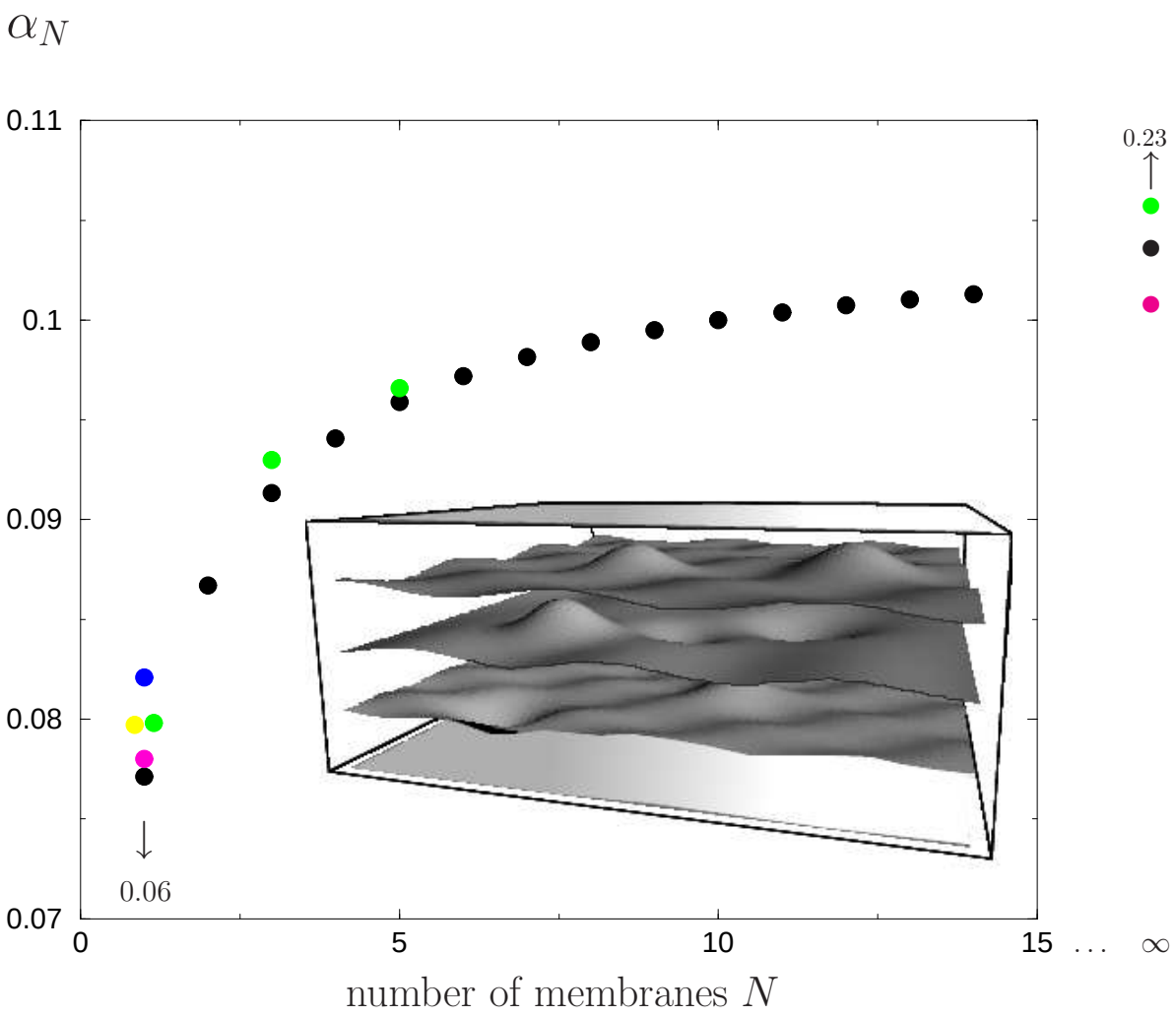
5-/6-Loop Result:

$\alpha_{\rm VPT}^{(5)} \approx 0.0820,$ $\alpha_{\rm VPT}^{(6)} = 0.0821 \pm 0.0005$

B. Kastening: Phys. Rev. **E 66**, 061102 (2002); in preparation

Stack of Membranes:

$$p = \alpha_N \frac{(k_B T)^2}{\kappa (d/2)^3}$$



[↑] W. Helfrich: Z. Naturforsch. **A 33**, 305 (1978)

[↓] W. Janke, H. Kleinert: Phys. Lett. **A 117**, 353 (1986)

[●] W. Janke, H. Kleinert, M. Meinhart: Phys. Lett. **B 217**, 525 (1989)

[●] G. Gompper, D.M. Kroll: Europhys. Lett. **9**, 59 (1989)

[●] M. Bachmann, H. Kleinert, A. Pelster: Phys. Lett. **A 261**, 127 (1999)

[●] M. Bachmann, H. Kleinert, A. Pelster: Phys. Rev. **E 63**, 051709 (2001)

[●] B. Kastening: Phys. Rev. **E 66**, 061102 (2002); in preparation

2.2 Quantum Statistics

Partition Function:

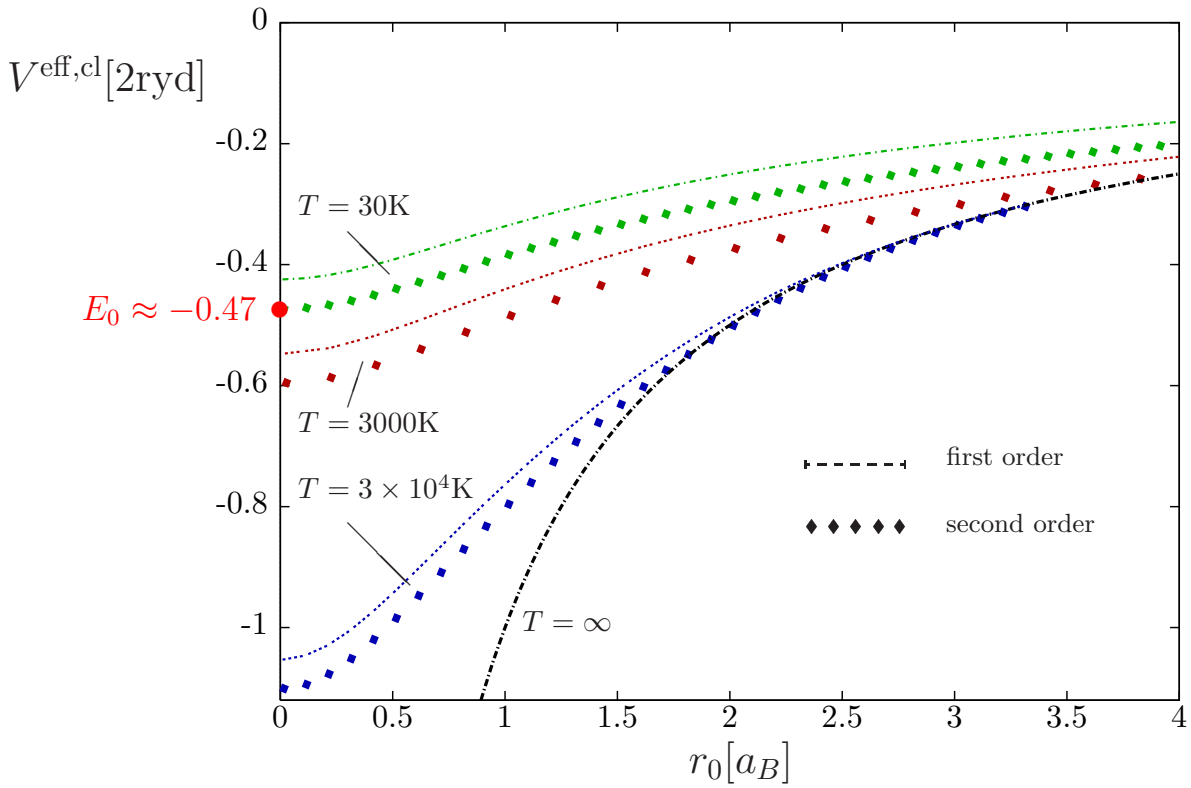
$$Z = \left(\frac{M k_B T}{2\pi\hbar^2}\right)^{3/2} \int d^3\mathbf{x_0} \ \exp\left[-\frac{V^{\mathrm{eff,cl}}(\mathbf{x_0})}{k_B T}\right]$$

R.P. Feynman, H. Kleinert: Phys. Rev. **A 34**, 5080 (1986)

Harmonic Trial System:

$$V_\Omega^{\mathbf{x}_0}(\mathbf{x}) = \frac{M}{2}\,\Omega^2(\mathbf{x_0})\,(\mathbf{x}-\mathbf{x_0})^2$$

Effective Classical Potential: Coulomb System

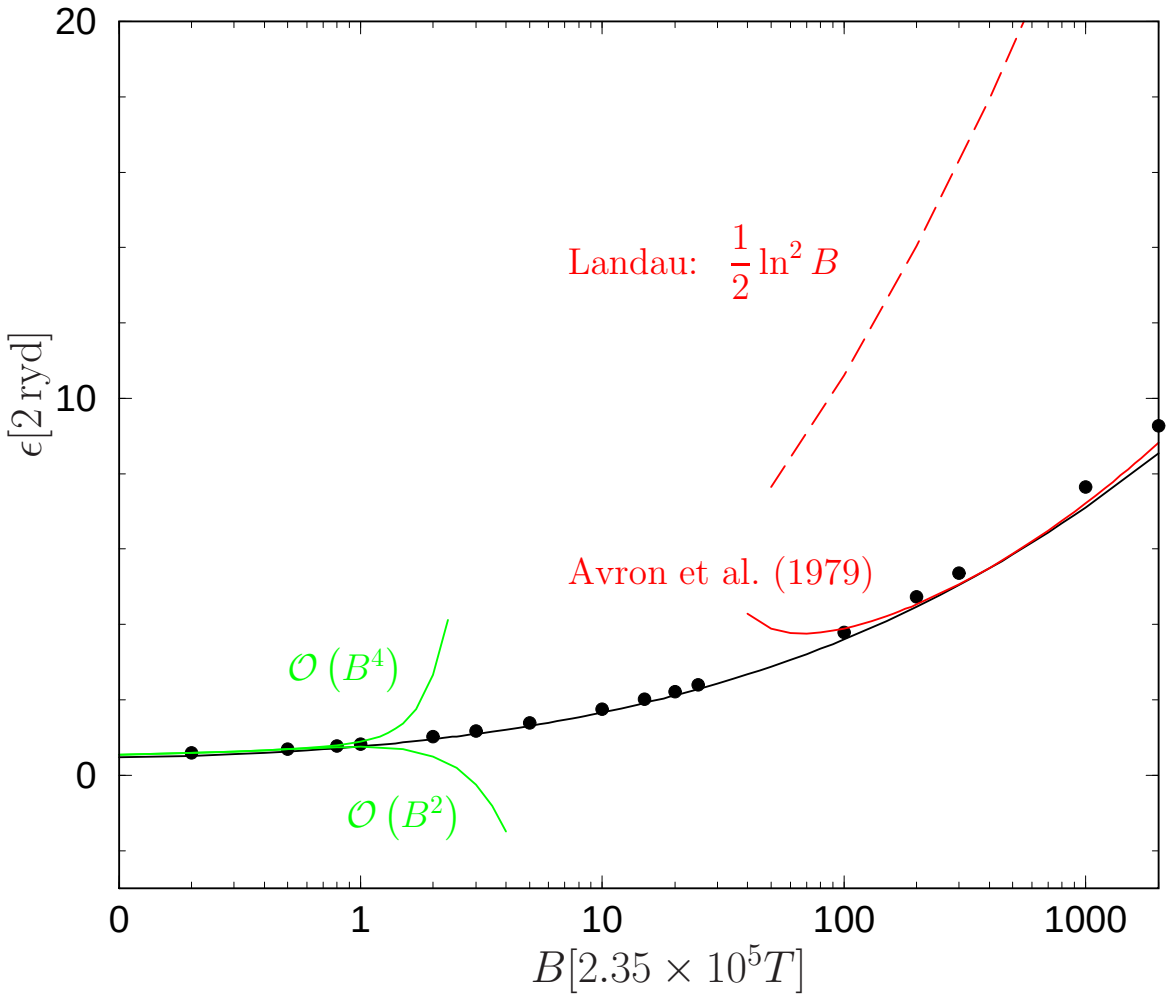


H. Kleinert, W. Kürzinger, A. Pelster: J. Phys. **A 31**, 8307 (1998)

Coulomb System With Magnetic Field (Without Spin):

$$H = \frac{1}{2M}\left(\mathbf{p} - \frac{e}{2}\mathbf{B}\times\mathbf{r}\right)^2 - \frac{e^2}{4\pi\epsilon_0|\mathbf{r}|}\,, \qquad \mathbf{B} = B\,\mathbf{e}_z$$

Binding Energy: $\epsilon(B) = B/2 - E_0(B)$



— : M. Bachmann, H. Kleinert, A. Pelster: Phys. Rev. **A 62**, 52509 (2000); Phys. Lett. **A 279**, 23 (2001)

•••• : J.C. Le Guillou, J. Zinn-Justin: Ann. Phys. **147**, 57 (1983); H. Ruder, G. Wunner, H. Herold, F. Geyer: Springer (1994)

2.3 Bose-Einstein Condensation

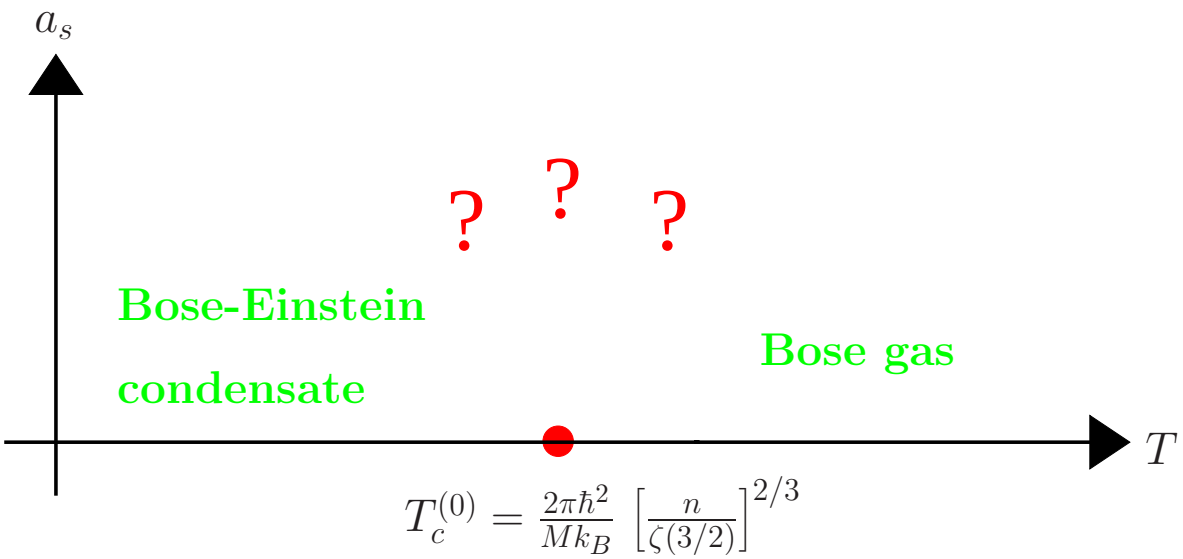
Model:

Bose gas with local repulsive 2-particle interaction

$$\mathcal{L}^{(3+1)} = \psi^* \left(\hbar\,\partial_\tau - \frac{\hbar^2}{2M}\,\mathbf{\Delta} - \mu \right) \psi + \frac{g}{2}\,|\psi|^4$$

s-wave scattering length: $\qquad a_s = \dfrac{M}{4\pi\hbar^2}\,g$

Goal: a_s-T-phase diagram



Methods:

- $T > T_c$: perturbation theory with respect to a_s
- $T < T_c$: loop expansion

$\Longrightarrow$ resummation via variational perturbation theory

High Temperature Approximation:

$$\mathcal{L}^{(3+1)} \Longrightarrow \mathcal{L}^{(3)} = \frac{1}{2}|\boldsymbol{\nabla}\psi_0|^2 + \frac{1}{2}\,m^2|\psi_0|^2 + \frac{u}{4!}|\psi_0|^4$$

Matsubara zero mode: ψ_0 ; $\quad m^2 \propto -\mu\,, \quad u \propto a_s$

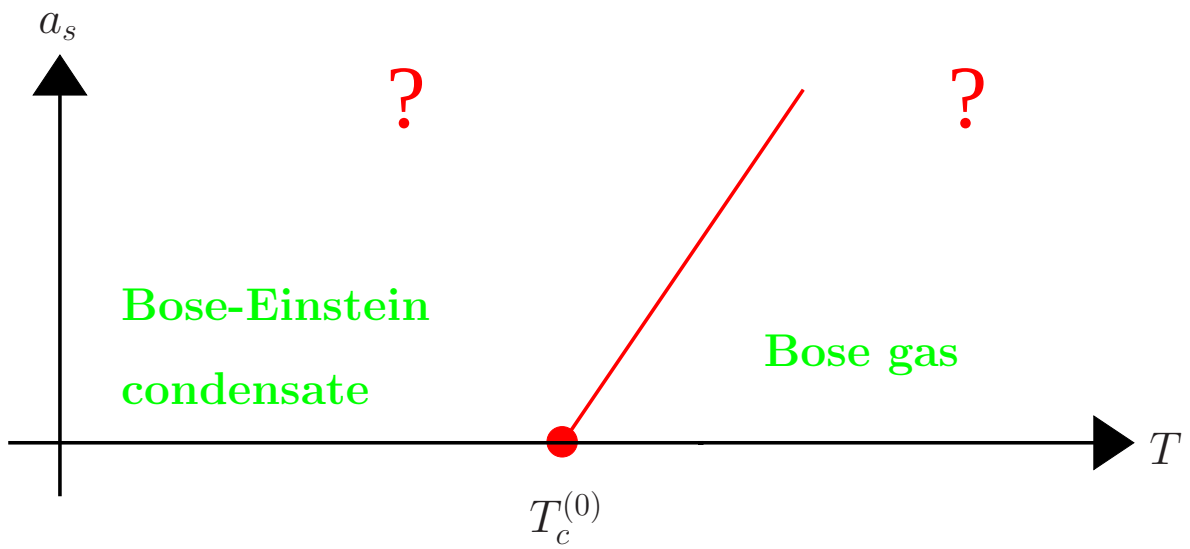
Critical Temperature for Dilute Bose Gas:

$$\frac{\Delta T_c}{T_c^{(0)}} \approx -\frac{2}{3}\frac{\Delta n}{n} \approx c\,a_s\,n^{1/3}\,, \qquad c^{(N)} = \sum_{k=1}^{N} c^{(k)} \left(\frac{u}{m_r}\right)^k$$

critical limit $m_r \longrightarrow 0 \quad \Longleftrightarrow$ strong-coupling problem

Results For c:

Kleinert (2002)	5-loop VPT	1.23 ± 0.17
Kastening (2003)	6-/7-loop VPT	1.27 ± 0.11
Svistunov et al. (2001)	Monte-Carlo	1.29 ± 0.05
Arnold, Moore (2001)	Monte-Carlo	1.32 ± 0.02



1-Loop Result: Effective Potential

$$\begin{aligned}\mathcal{V}(\Psi,\Psi^*) &= V\left(-\mu|\Psi|^2+\frac{g}{2}|\Psi|^4\right)\\ &\quad+\frac{1}{2}\sum_{\mathbf{k}}E(\mathbf{k})+k_BT\sum_{\mathbf{k}}\ln[1-e^{-E(\mathbf{k})/k_BT}]\\ E(\mathbf{k}) &= \sqrt{[\epsilon(\mathbf{k})-\mu+2g|\Psi|^2]^2-g^2|\Psi|^4}\end{aligned}$$

Extremization in Ψ, Ψ^*:

$$\begin{aligned}\text{condensate density:}\quad n_0(\mu,T) &= \Psi^*\Psi\\ \text{grand-canonical potential:}\quad \Omega(\mu,T) &= \mathcal{V}(\Psi,\Psi^*)\\ \text{particle density:}\quad n(\mu,T) &= -\frac{1}{V}\frac{\partial\Omega(\mu,T)}{\partial\mu}\end{aligned}$$

Elimination of μ and VPT Resummation:

$$\begin{aligned}n &= n_0+\frac{1}{V}\sum_{\mathbf{k}}\frac{\epsilon(\mathbf{k})+gn}{\sqrt{\epsilon(\mathbf{k})^2+2gn\epsilon(\mathbf{k})}}\\ &\quad\times\left(\frac{1}{2}+\frac{1}{e^{\sqrt{\epsilon(\mathbf{k})^2+2gn\epsilon(\mathbf{k})}/k_BT}-1}\right)\end{aligned}$$

Dispersion: $\epsilon(\mathbf{k}) = \dfrac{\hbar^2 \mathbf{k}^2}{2M}\,, \qquad \displaystyle\sum_{\mathbf{k}} = V \int \frac{d^3k}{(2\pi)^3}$

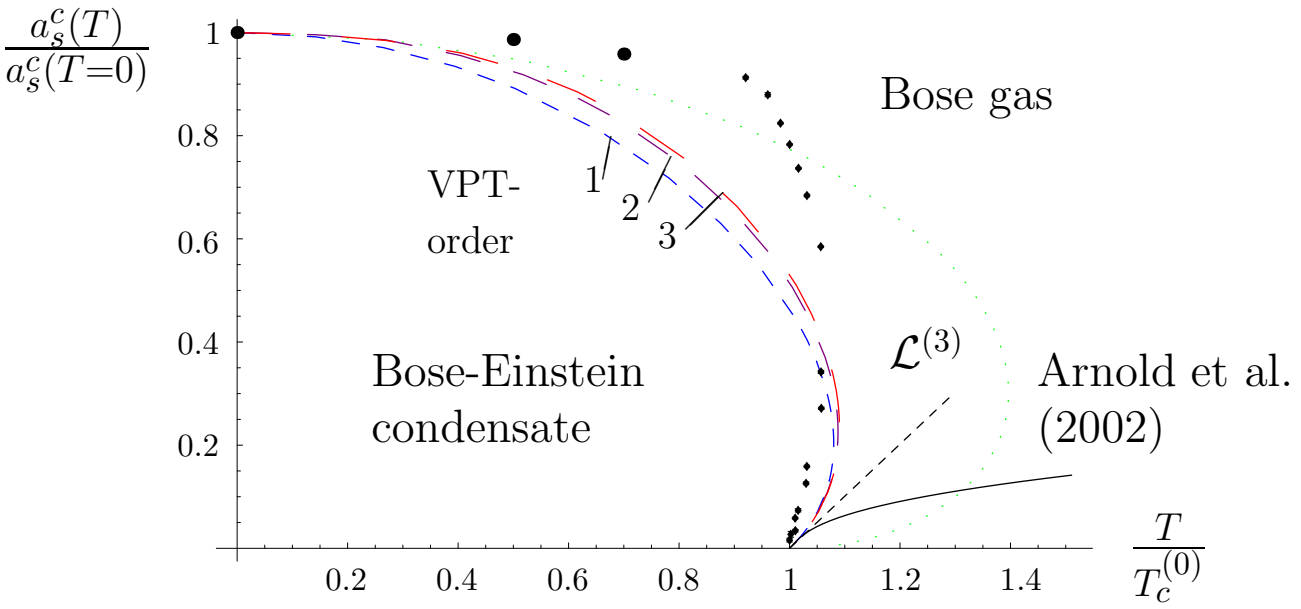
Phase Transition: $n_0 \equiv 0$

$$\left(\frac{9\pi}{64}\right)^{1/3} = a_s n^{1/3} \left[1 + \frac{3\alpha}{16} I(\alpha)\right]^{2/3}$$

$$I(\alpha) = \int_0^\infty dx \, \frac{x\alpha + 8}{2\sqrt{x\alpha + 16}\,\left(e^{\sqrt{x + x^2\alpha/16}} - 1\right)}$$

$$\alpha = \left(\frac{T}{a_s\, n^{1/3} \zeta(3/2)^{2/3}\, T_c^{(0)}}\right)^2$$

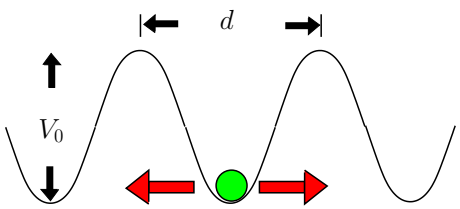
Calculated Phase Diagram:



H. Kleinert, S. Schmidt, A. Pelster: `cond-mat/0307412`

BEC in Optical Lattice:

$$V(\mathbf{x}) = V_0 \sum_{i=1}^{3} \sin^2\left(\frac{\pi x_i}{d}\right)$$



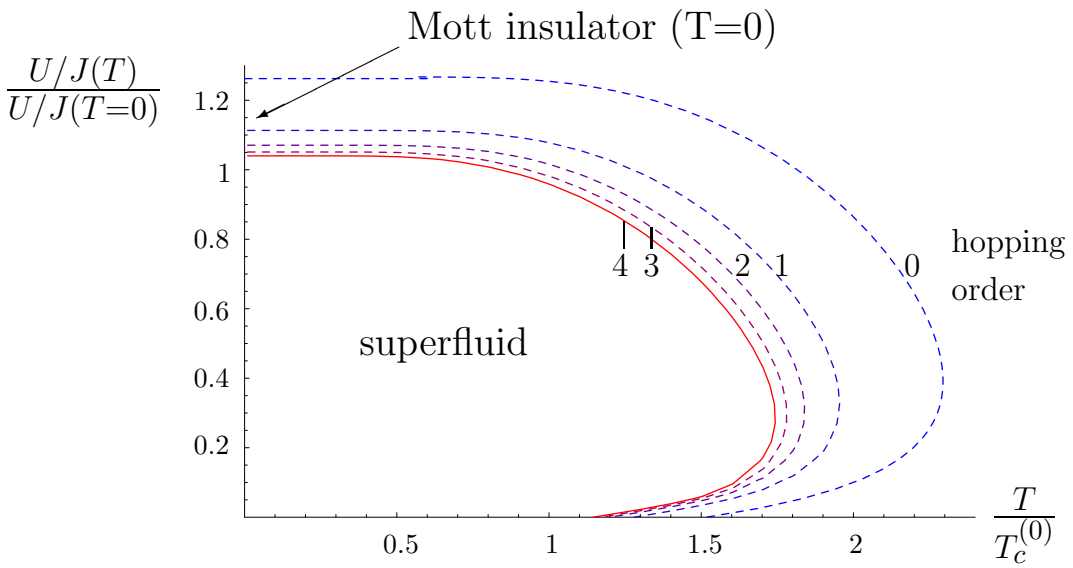
Dispersion:

$$\epsilon(\mathbf{k}) = 2J \sum_{i=1}^{3} (1 - \cos k_i d)\,, \qquad \sum_{\mathbf{k}} = V \prod_{i=1}^{3} \int_{-\pi/d}^{\pi/d} \frac{dk_i}{2\pi}$$

Phase Transition: $n_0 \equiv 0$

$$F\left(\frac{k_B T}{J}, \frac{U}{J}\right) \quad = \quad 0\,, \qquad U = g\,n$$

Calculated Phase Diagram:



experiment (Greiner et al.: 2002): $U/J(T=0) \approx 33-36$

quantum phase transition: $U/J(T=0) \approx 31$

H. Kleinert, S. Schmidt, A. Pelster: `cond-mat/0307412`

2.4. Markov Theory

Fokker-Planck Equation:

$$\frac{\partial}{\partial t}P(x,t) \;=\; \left[-\frac{\partial}{\partial x}K(x) + D\,\frac{\partial^2}{\partial x^2}\right] P(x,t)$$

Stochastic Model: Laser

$$K(x) \quad = \quad -\gamma x - g x^3$$

x: electric field strength

Perturbation Theory:

$$P(x,t) = P_\gamma(x,t) \sum_{n=0}^{\infty} g^n \sum_{k=0}^{4n} C_{n,k}(t) x^k$$

J. Dreger, B. Hamprecht, A. Pelster: in preparation

Identity:

$$K(x) = \underbrace{-\kappa x}_{\text{effective Brownian particle}} - \underbrace{\left[g x^3 + (\gamma - \kappa)\; x\right]}_{\text{perturbation}}$$

κ: variational parameter

H. Kleinert, A. Pelster, M.V. Putz: Phys. **E 65**, 066128 (2002)

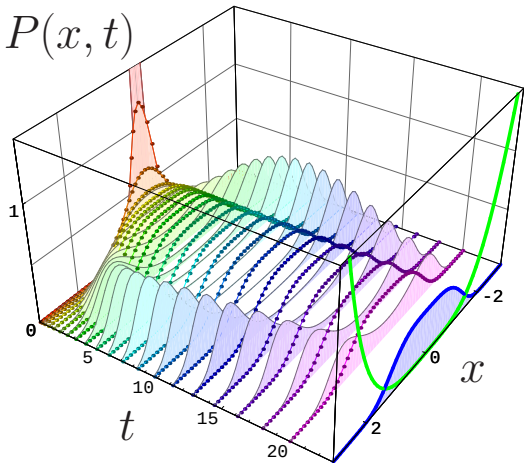
Principle of Minimal Sensitivity:

$$\frac{\partial P_\kappa(x,t)}{\partial \kappa} = 0 \quad \Longrightarrow \quad \kappa = \kappa(x,t)$$

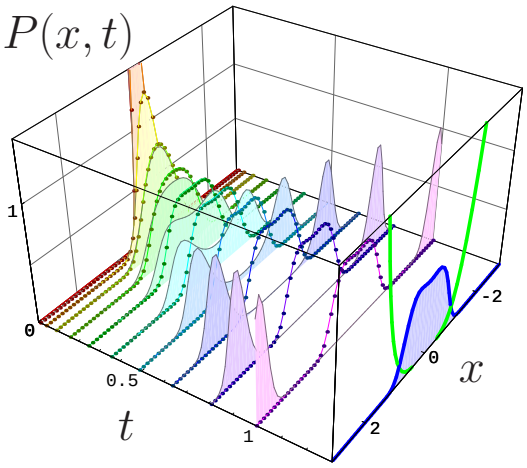
$$P(x,t) \approx \frac{P_{\kappa(x,t)}(x,t)}{\int_{-\infty}^{+\infty} dx\, P_{\kappa(x,t)}(x,t)}$$

Results in First Order:

below laser instability:
$g = 1, \gamma = 0.1, D = 1$



above laser instability:
$g = 10, \gamma = -1, D = 1$



J. Dreger, B. Hamprecht, A. Pelster: in preparation

3.1. Further Applications

Quantum Statistics:

- higher orders

  F. Weißbach, A. Pelster, B. Hamprecht: Phys. Rev. **E 66**, 036129 (2002)

- density matrices

  M. Bachmann, H. Kleinert, A. Pelster: Phys. Rev. **A 60**, 3429 (1999)

- tunneling amplitudes

  B. Hamprecht, H. Kleinert: `hep-th/0302124`

Nonlinear Dynamical Systems:

A. Pelster, H. Kleinert, M. Schanz: Phys. Rev. **E 67**, 016604 (2003)

Boundary Layers:

C.M. Bender, A. Pelster, F. Weißbach: J. Math. Phys. **43**, 4202 (2002)

Critical Exponents:

H. Kleinert, V. Schulte-Frohlinde: *Critical Properties of* ϕ^4*-Theories*, World Scientific (2001)

3.2. Further Problems

Development of VPT:

- several coupling constants
- confluent singularities

Soft Matter:

adhesion of vesicles on surfaces

$$E = \int d^2x \left\{ \frac{\kappa}{2} \left[\boldsymbol{\nabla}^2 \varphi(\mathbf{x}) \right]^2 + \nu \left[\boldsymbol{\nabla} \varphi(\mathbf{x}) \right]^2 \right\}$$

U. Seifert: Phys. Rev. Lett. **74**, 5060 (1995)

Bose-Einstein Condensation:

- influence of trap on phase diagram
- magnetic dipole-dipole interaction

Time-Delayed Systems:

$$\dot{q}(t) = N\left(q(t), q(t-\tau)\right)$$

- C. Simmendinger, A. Pelster, A. Wunderlin: Phys. Rev. **E 59**, 5344 (1999)
- M. Schanz, A. Pelster: Phys. Rev. **E 67**, 056205 (2003)
- M. Schanz, A. Pelster: SIAM Journal on Applied Dynamical Systems (in press)