

Two-Fluid Model for Anisotropic Superfluids

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Introduction

Hydrodynamic Theory of Superfluidity

- 1937 Kapitza: superfluid helium
- strange behavior:
 - ▶ zero viscosity for capillary flow
 - ▶ finite viscosity for rotating flow
- 1941 Landau – Landau-Khalatnikov Two-Fluid Model
- generic equations: application in ultracold quantum gases
- this talk: generalization to dipolar gases

Outline

- Derivation of Landau-Khalatnikov Two-Fluid Equations
 - ▶ Action Principle
 - ▶ Classic Approach (Khalatnikov)
- Sound Propagation
- Generalization to Anisotropic Systems

Assumption I – Conservation of Mass

- mass density

$$\rho = \rho_s + \rho_n$$

- mass current

$$\mathbf{j} = \rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s$$

- continuity equation

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \mathbf{j} = 0$$

Assumption II – Conservation of Entropy

- entropy density

$$s = s_n, s_s = 0$$

- entropy conservation

$$\frac{\partial s}{\partial t} + \operatorname{div} s \mathbf{v}_n = 0$$

Action Principle

$$\begin{aligned} \mathcal{A} = \int d\mathbf{r} dt & \left\{ \frac{1}{2}(\rho - \rho_n)\mathbf{v}_s^2 + \frac{1}{2}\rho_n\mathbf{v}_n^2 - u(\rho, \rho_n, s) \right. \\ & \left. + \lambda \left[\frac{\partial \rho}{\partial t} + \operatorname{div}((\rho - \rho_n)\mathbf{v}_s + \rho_n\mathbf{v}_n) \right] + \kappa \left[\frac{\partial s}{\partial t} + \operatorname{div}(s\mathbf{v}_n) \right] \right\} \end{aligned}$$

Differential equation of state

$$du = Tds + \bar{\mu}d\rho + \frac{\partial u}{\partial \rho_n} d\rho_n , \quad \bar{\mu} = \mu/m$$

Demand

$$\frac{\delta \mathcal{A}}{\delta \rho}, \frac{\delta \mathcal{A}}{\delta \rho_n}, \frac{\delta \mathcal{A}}{\delta s}, \frac{\delta \mathcal{A}}{\delta \mathbf{v}_s}, \frac{\delta \mathcal{A}}{\delta \mathbf{v}_n} \stackrel{!}{=} 0$$

Equations

$$\frac{\delta \mathcal{A}}{\delta \rho} : \quad \frac{1}{2} \mathbf{v}_s^2 - \bar{\mu} - \frac{\partial \lambda}{\partial t} - \mathbf{v}_s \cdot \nabla \lambda = 0 , \quad (1)$$

$$\frac{\delta \mathcal{A}}{\delta \rho_n} : \quad \frac{1}{2} \mathbf{v}_n^2 - \frac{1}{2} \mathbf{v}_s^2 - \frac{\partial u}{\partial \rho_n} + (\mathbf{v}_s - \mathbf{v}_n) \cdot \nabla \lambda = 0 , \quad (2)$$

$$\frac{\delta \mathcal{A}}{\delta s} : \quad - T - \frac{\partial \kappa}{\partial t} - \mathbf{v}_n \cdot \nabla \kappa = 0 , \quad (3)$$

$$\frac{\delta \mathcal{A}}{\delta \mathbf{v}_s} : \quad (\rho - \rho_n)(\mathbf{v}_s - \nabla \lambda) = \mathbf{0} , \quad (4)$$

$$\frac{\delta \mathcal{A}}{\delta \mathbf{v}_n} : \quad \rho_n(\mathbf{v}_n - \nabla \lambda) - s \nabla \kappa = \mathbf{0} . \quad (5)$$

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Consequences

- irrotational superfluid flow

$$\mathbf{v}_s = \nabla \lambda \quad \Rightarrow \quad \text{rot } \mathbf{v}_s = \mathbf{0}$$

- superfluid Euler equation

$$\frac{\partial \mathbf{v}_s}{\partial t} + (\mathbf{v}_s \cdot \nabla) \mathbf{v}_s = -\nabla \bar{\mu}$$

- completion of differential state equation

$$\frac{\partial u}{\partial \rho_n} = \frac{1}{2}(\mathbf{v}_n - \mathbf{v}_s)^2$$

Consequences

- normalfluid Euler equation

$$\frac{\partial \mathbf{v}_n}{\partial t} + (\mathbf{v}_n \cdot \nabla) \mathbf{v}_n = -\frac{s}{\rho_n} \nabla T - \nabla \bar{\mu} - \nabla \frac{1}{2} (\mathbf{v}_n - \mathbf{v}_s)^2 - \frac{\Gamma}{\rho_n} (\mathbf{v}_n - \mathbf{v}_s)$$

- source term

$$\Gamma = \frac{\partial \rho_n}{\partial t} + \text{div}(\rho_n \mathbf{v}_n)$$

Result

- reformulation in terms of mass current

$$\frac{\partial j_i}{\partial t} + \frac{\partial \Pi_{ik}}{\partial x_k} = 0$$

- momentum density tensor

$$\Pi_{ik} = \rho_s v_{si} v_{sk} + \rho_n v_{ni} v_{nk} + p \delta_{ik} ,$$

- pressure

$$p = T s + \bar{\mu} \rho + \frac{1}{2} (\mathbf{v}_n - \mathbf{v}_s)^2 \rho_n - u$$

Concept of Classical Approach

- state formal conservation laws, e.g.

$$\frac{\partial \mathbf{v}_s}{\partial t} + (\mathbf{v}_s \cdot \nabla) \mathbf{v}_s = -\nabla \varphi$$
$$\frac{\partial s}{\partial t} + \operatorname{div} \mathbf{f} = 0$$

- Galilean transformation to reference system $\mathbf{v}_{s,0} = \mathbf{0}$, e.g.

$$\Pi_{ik} = \Pi_{0ik} + \rho v_{s_i} v_{s_k} + v_{s_i} j_{0k} + j_{0i} v_{s_k}$$

- differential equation of state in superfluid rest-frame

$$d\epsilon_0 = T ds + \bar{\mu} d\rho + \mathbf{v}_{n_0} d\mathbf{j}_0$$

Goal

- no dissipation \Rightarrow energy flux \mathbf{q}_0 has certain structure, i.e.
no dependence on 'gradient terms'
- find an expression for $\operatorname{div} \mathbf{q}_0$
- read off constraints from the expression $\operatorname{div} \mathbf{q}_0$

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$$\begin{aligned}\operatorname{div} \mathbf{q}_0 = & -(\mathbf{h} \cdot \nabla) \cdot \mathbf{v}_s + (\mathbf{v}_n - \mathbf{v}_s) \cdot (\nabla \cdot \mathbf{h}) + \mathbf{j}_0 \cdot [(\mathbf{v}_n - \mathbf{v}_s) \cdot \nabla] \mathbf{v}_n \\ & - \rho_s (\mathbf{v}_n - \mathbf{v}_s) \cdot \nabla (\varphi - \bar{\mu}) - \nabla T \cdot [\mathbf{f}_0 - s(\mathbf{v}_n - \mathbf{v}_s)] \\ & + \operatorname{div}(\mathbf{f}_0 T + \mathbf{j}_0 \bar{\mu})\end{aligned}$$

$$\mathbf{h}_{ik} = \Pi_{0_{ik}} + [e_0 - Ts - \bar{\mu}\rho - (\mathbf{v}_n - \mathbf{v}_s) \cdot \mathbf{j}_0] \delta_{ik}$$

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- e.g. entropy flux $\quad \mathbf{f}_0 = s(\mathbf{v}_n - \mathbf{v}_s) \quad \Rightarrow \quad \mathbf{f} = s\mathbf{v}_n$

Comparison

Similarities

- lead to the same equations
- based on the existence of conservation laws

Differences

- Action Principle: $s_s = 0 \Rightarrow \text{rot } \mathbf{v}_s = 0$
- Khalatnikov Approach: $\text{rot } \mathbf{v}_s = 0 \Rightarrow s_s = 0$
- $s_s = 0 \Leftrightarrow \text{rot } \mathbf{v}_s = 0$ shown for action principle
- extending and varying the system much easier for action principle, e.g. include superfluid vorticity

Linearization

Ansatz: $\rho(\mathbf{r}, t) = \rho_0 + \delta\rho(\mathbf{r}, t)$ etc.

$$\frac{\partial \delta\rho}{\partial t} + \operatorname{div} \delta\mathbf{j} = 0$$

$$\delta\mathbf{j} = \rho_{s0}\delta\mathbf{v}_s + \rho_{n0}\delta\mathbf{v}_n$$

$$\frac{\partial \delta s}{\partial t} + s_0 \operatorname{div} \delta\mathbf{v}_n = 0$$

$$\frac{\partial \delta\mathbf{v}_s}{\partial t} = -\nabla \delta\bar{\mu}$$

$$\frac{\partial \delta\mathbf{j}}{\partial t} = -\nabla \delta p$$

Derivation of Wave Equations

$$\Rightarrow \frac{\partial^2 \delta \rho}{\partial t^2} - \Delta \delta p = 0$$

$$\frac{\partial \delta \bar{s}}{\partial t} = \frac{s_0}{\rho_0^2} \rho_{s_0} \operatorname{div}(\delta \mathbf{v}_n - \delta \mathbf{v}_s), \quad \bar{s} = s/\rho$$

$$\Rightarrow \frac{\partial^2 \delta \bar{s}}{\partial t^2} - \bar{s}_0^2 \frac{\rho_{s_0}}{\rho_{n_0}} \Delta \delta T = 0$$

$$\delta p = \left(\frac{\partial p}{\partial \rho} \right)_{\bar{s}} \delta \rho + \left(\frac{\partial p}{\partial \bar{s}} \right)_{\rho} \delta \bar{s}, \quad \delta T = \left(\frac{\partial T}{\partial \rho} \right)_{\bar{s}} \delta \rho + \left(\frac{\partial T}{\partial \bar{s}} \right)_{\rho} \delta \bar{s}$$

Coupled Wave Equations

$$\frac{\partial^2 \delta\bar{s}}{\partial t^2} - \bar{s}_0^2 \frac{\rho_{s_0}}{\rho_{n_0}} \left[\left(\frac{\partial T}{\partial \rho} \right)_{\bar{s}} \Delta \delta\rho + \left(\frac{\partial T}{\partial \bar{s}} \right)_{\rho} \Delta \delta\bar{s} \right] = 0$$

$$\frac{\partial^2 \delta\rho}{\partial t^2} - \left[\left(\frac{\partial p}{\partial \rho} \right)_{\bar{s}} \Delta \delta\rho + \left(\frac{\partial p}{\partial \bar{s}} \right)_{\rho} \Delta \delta\bar{s} \right] = 0$$

plane waves $\delta\rho, \delta\bar{s} \propto e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)}$ with speed of sound $u = \omega/q$

$$u^4 - u^2 \left[\left(\frac{\partial p}{\partial \rho} \right)_{\bar{s}} + \frac{\rho_{s_0} s_0^2}{\rho_{n_0}} \left(\frac{\partial T}{\partial \bar{s}} \right)_{\rho} \right] + \bar{s}_0^2 \frac{\rho_{s_0}}{\rho_{n_0}} \left[\left(\frac{\partial T}{\partial \bar{s}} \right)_{\rho} \left(\frac{\partial p}{\partial \rho} \right)_{\bar{s}} - \left(\frac{\partial p}{\partial \bar{s}} \right)_{\rho} \left(\frac{\partial T}{\partial \rho} \right)_{\bar{s}} \right] = 0$$

First and Second Sound

- approximation for a liquid $\left(\frac{\partial p}{\partial T}\right)_\rho \simeq 0$

- first sound

- ▶ $u_1^2 = \left(\frac{\partial p}{\partial \rho}\right)_{\bar{s}}$
- ▶ pressure or density wave \Leftrightarrow usual sound wave
- ▶ in-phase motion $\delta \mathbf{v}_s = \delta \mathbf{v}_n$

- second sound

- ▶ $u_2^2 = \bar{s}_0^2 \frac{\rho_{s0}}{\rho_{n0}} \left(\frac{\partial T}{\partial \bar{s}}\right)_\rho$
- ▶ temperature or entropy wave \Leftrightarrow unique for superfluids
- ▶ out-of-phase motion $\rho_{s0} \delta \mathbf{v}_s = -\rho_{n0} \delta \mathbf{v}_n$

Anisotropic Extension

Motivation

- fully polarized dipolar BEC at finite temperatures
- fully polarized dipolar BEC at $T = 0K$ with disorder^{1,2}

$$n_{sik} = n\delta_{ik} - \int \frac{d^3k}{(2\pi)^2} \frac{4nR(\mathbf{k})k_ik_k}{\mathbf{k}^2 [\hbar^2\mathbf{k}^2/2m + 2nV_{\text{int}}(\mathbf{k})]^2} + \dots$$

- superfluid density has the structure of a rank two tensor
- system's response depends on the direction of perturbation

¹C. Krumnow and A. Pelster *Phys. Rev. A* **84** 021608 (2011)

²B. Nikolic, A. Balaz and A. Pelster, arXiv:1212.4807

Anisotropic Extension

Hydrodynamic equations for anisotropic fluid

- total density remains scalar $\rho_{n_{ik}} + \rho_{s_{ik}} = \rho_{ik} = \tilde{\rho}\delta_{ik}$

- continuity equation $\partial_t \tilde{\rho} + \operatorname{div} \mathbf{j} = 0$

- mass current $j_i = \rho_{n_{ij}} v_{n_j} + \rho_{s_{ij}} v_{s_j}$

- momentum conservation $\frac{\partial j_i}{\partial t} + \partial_k \Pi_{ik} = 0$

- $\Pi_{ik} = \rho_{s_{kj}} v_{s_j} v_{s_i} + \rho_{n_{ij}} v_{n_j} v_{n_k} - \rho_{n_{ij}} v_{n_k} v_{s_j} + \rho_{n_{kj}} v_{n_j} v_{s_i} + p\delta_{ik}$

- momentum tensor: no symmetric structure $\Pi_{ik} \neq \Pi_{ki}$

Asymmetry of momentum tensor

Relation to the conservation of angular momentum

- usual hydrodynamics

- ▶ symmetric momentum tensor
- ▶ consequence of conservation of angular momentum
- ▶ consequence of isotropy of space (Noether)

- anisotropic fluid

- ▶ broken symmetry
- ▶ angular momentum needs not be conserved
- ▶ no need for symmetric momentum tensor

First and Second Sound in an anisotropic superfluid

- coupled wave equations

$$\frac{\partial^2 \delta\tilde{\rho}}{\partial t^2} - \left[\left(\frac{\partial p}{\partial \tilde{\rho}} \right)_{\bar{s}} \Delta \delta\tilde{\rho} + \left(\frac{\partial p}{\partial \bar{s}} \right)_{\tilde{\rho}} \Delta \delta\bar{s} \right] = 0$$

$$\frac{\partial^2 \delta\bar{s}}{\partial t^2} - \bar{s}_0^2 \rho_{n0ik}^{-1} \rho_{s0kj} \left[\left(\frac{\partial T}{\partial \tilde{\rho}} \right)_{\bar{s}} \partial_i \partial_j \delta\tilde{\rho} + \left(\frac{\partial T}{\partial \bar{s}} \right)_{\tilde{\rho}} \partial_i \partial_j \delta\bar{s} \right] = 0$$

- ansatz $\delta\bar{s}, \delta\tilde{\rho} \propto e^{i(\mathbf{q} \cdot \mathbf{r} - \omega t)}$, $\mathbf{q}_{\perp} \perp \hat{\mathbf{e}}_{\text{dipole}}$, $\mathbf{q}_{\parallel} \parallel \hat{\mathbf{e}}_{\text{dipole}}$

- diagonal form $\rho_n = \begin{pmatrix} \rho_{n0\perp} & 0 & 0 \\ 0 & \rho_{n0\perp} & 0 \\ 0 & 0 & \rho_{n0\parallel} \end{pmatrix}$

Sound velocities

- sound velocities

$$u_{1,2\parallel,\perp}^2 = \frac{1}{2} \left[\left(\frac{\partial p}{\partial \tilde{\rho}} \right)_{\bar{s}} + \frac{\rho_{s_0\parallel,\perp}}{\rho_{n_0\parallel,\perp}} \frac{T \bar{s}_0^2}{c_\nu} \right] \\ \pm \sqrt{\frac{1}{4} \left[\left(\frac{\partial p}{\partial \tilde{\rho}} \right)_{\bar{s}} + \frac{\rho_{s_0\parallel,\perp}}{\rho_{n_0\parallel,\perp}} \frac{T \bar{s}_0^2}{c_\nu} \right]^2 - \frac{\rho_{s_0\parallel,\perp}}{\rho_{n_0\parallel,\perp}} \left(\frac{T \bar{s}_0^2}{c_\nu} \right) \left(\frac{\partial p}{\partial \tilde{\rho}} \right)_T}$$

- expression remains complicated, because we do not assume, that pressure and temperature fluctuations are uncoupled $\left(\frac{\partial p}{\partial T} \right)_\rho \neq 0$

Eigenmodes and Associated Motions

- modes do not decouple

- ▶ $\frac{\delta\tilde{\rho}}{\delta\tilde{s}} = \frac{\left(\frac{\partial p}{\partial\tilde{s}}\right)_{\tilde{\rho}}}{u_{\parallel,\perp}^2 - \left(\frac{\partial p}{\partial\tilde{\rho}}\right)_{\tilde{s}}} = r$

- ▶ r coupling strength of the two modes

- motion along the direction of wave propagation

- ▶ $\delta\mathbf{v}_n = \nu\delta\mathbf{v}_s \parallel \hat{\mathbf{e}}_{\mathbf{q}_{\parallel,\perp}}$

- ▶ ν determines in-phase/out-of-phase motion

- ▶ $\nu = \frac{(\alpha - \beta\tilde{s}_0\tilde{\rho}_0)\rho n_0_{\parallel,\perp}}{\tilde{\rho}_0\alpha - \rho s_0_{\parallel,\perp}(\alpha - \tilde{\rho}_0\tilde{s}_0\beta)}$

- ▶ $\alpha = \left(\frac{\partial p}{\partial\tilde{\rho}}\right)_{\tilde{s}} r + \left(\frac{\partial p}{\partial\tilde{s}}\right)_{\tilde{\rho}}, \beta = \left(\frac{\partial T}{\partial\tilde{\rho}}\right)_{\tilde{s}} r + \left(\frac{\partial T}{\partial\tilde{s}}\right)_{\tilde{\rho}}$

Conclusion and Outlook

- calculate equation of state e.g. for a dipolar Bose gas
- generalization to non-fully polarized dipolar gas³
- similar, but much more extensive work in the context of Helium-3⁴
- extension to three-fluid model (localized BECs)
- experimental verification of results

³G. Szirmai and P. Szépfalusy, *Phys. Rev. A* **85** 053603 (2012)

⁴D. Vollhardt and P. Wölfle, *The Superfluid Phases of Helium 3*,

Taylor & Francis 1990