Universal Spectral Statistics in Quantum Graphs

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We prove that the spectrum of an individual chaotic quantum graph shows universal spectral correlations, as predicted by random-matrix theory. The stability of these correlations with regard to nonuniversal corrections is analyzed in terms of the linear operator governing the classical dynamics on the graph.

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Fluctuations in spectra of individual complex quantum systems (e.g., classically chaotic systems) are universal and can be described by the Gaussian ensembles of random-matrix theory (RMT). This statement, promoted to a conjecture by Bohigas, Giannoni, and Schmit [1], has been empirically confirmed in numerous experimental and numerical analyses [2–4]. However, never so far has it been possible to demonstrate analytically that spectral fluctuations of *individual* [5] chaotic systems obey RMT statistics. Important progress has recently been made within the framework of periodic-orbit theory: summing over orbit pairs of nearly identical action it became possible to prove universality (agreement with the predictions of RMT) first for a few [7,8] and then all [9] coefficients of the *short time* expansion of the spectral form factor. Unfortunately it is not known how to advance this semiclassical approach into the regime of times larger than the Heisenberg time $t_H = \frac{2\pi\hbar}{\Delta E}$ (ΔE is the mean level spacing).

Here, we apply different theoretical concepts to prove universality (including the long time regime) for a family of chaotic quantum systems, the so-called quantum graphs [10]. Quantum graphs differ from generic Hamiltonian systems in that they are semiclassically exact (the density of states can be represented in terms of an exact semiclassical trace formula), while they do not possess an underlying deterministic classical dynamics. Still they display much of the behavior of generic hyperbolic quantum systems; equally important, they are not quite as resistant to analytical approaches as these.

In previous work, Berkolaiko *et al.* [11] developed a perturbative diagrammatic language to analyze the semiclassical periodic-orbit representation of spectral correlation functions beyond the leading ("diagonal") approximation. In spite of the full knowledge of its building blocks [11,12] a complete resummation of the perturbation series has so far been elusive (not to mention that such expansions are subject to the same limitations as the semiclassical approaches mentioned above). In contrast, our present approach avoids diagrammatic resummations. We rather build on two alternative pieces of input, both of which have been discussed separately before: (i) the exact equivalence of a spectral average for a quantum graph with incommensurate bond lengths to an average over a certain ensemble of unitary matrices [10,13,14] and (ii) the so-called color-flavor transformation [15], which is an exact mapping of the phase-averaged spectral correlation function onto a variant of the supersymmetric σ model. A subsequent stationary phase analysis then leads to the RMT correlation function corresponding to the symmetry of the graph. Finally, the spectrum of the "massive" fluctuations around the saddle point contains quantitative information on the stability of RMT spectral statistics with regard to nonuniversal corrections. Deferring the discussion of other symmetry classes to a separate publication [16], we consider graphs which are invariant under both time reversal and spin rotation.

Let us begin by introducing our basic setting. A quantum graph consists of V vertices j connected by B bonds b. For the topology of the graph we assume that pairs of vertices are connected by at most one bond and that no bond starts and ends at the same vertex [17]. We introduce $2B$ double indices (b, d), where $d = 1, 2$ determines the (arbitrarily defined) direction of propagation along $b = 1, \ldots, B$. Boundary conditions on the graph are set by the fixed $2B$-dimensional unitary matrix $S_{bd,b'd'}$ which describes the scattering of an incoming wave function on bond b to an outgoing wave function on bond b'. Of course, $S_{bd,b'd'}$ is nonvanishing only for bonds b and b' connecting at a common vertex j. Time-reversal invariance ($\mathcal{T}$ invariance) implies that $S^T = \sigma_1^{\rm dir} S \sigma_1^{\rm dir}$, where $\sigma_i^{\rm dir} = (\sigma_i^{\rm dir})_{dd'}$ are Pauli matrices in the space of directional indices. The complete dynamical information on the graph is carried by the $2B \times 2B$ bond scattering matrix $S(k) = T(k) S T(k)$. Here, the diagonal matrices $T(k)$ contain the dynamical quantum phases picked up during propagation at fixed wave number k along the bonds: $T(k)_{bb',dd'} = \delta_{bb'}\delta_{dd'} \exp i\frac{kL_b}{2}$, where L_b is the length of bond b and the twofold replication in direction space expresses the independence of the dynamical phases on the direction of propagation. The concise formulation of this fact reads as $\mathcal{T}: S(k) = \sigma_1^{\rm dir} S^T(k) \sigma_1^{\rm dir}$. The prime signature of chaotic dynamics on the graph are strong non-Poissonian correlations in its discrete spectrum $\{k_n\}$. The latter is defined by the condition that $S(k_n)$ has a unit eigenvalue or, equivalently, by the vanishing of $\xi(k) \equiv \det[1 - S(k)]$ at $k = k_n$. (This condition is equivalent [10] to the existence of an eigenvalue of the bond Schrödinger operator canonically associated to the scattering matrix. In this sense the spectrum $\{k_n\}$ is analogous to the discrete energy spectrum of a Hamiltonian chaotic system.) Below, we explore the two-point spectral correlation function $R_2(s) \equiv \Delta^2 \langle \rho(k + s\Delta)\rho(k)\rangle_k - 1$, where $\rho(k) \equiv \sum_n \delta(k - k_n)$ is the spectral density and $\langle \cdots \rangle_k \equiv \lim_{K\to\infty} \frac{1}{K} \int_0^K dk(\cdots)$ is an average over the wave number parameter k. ($\Delta = \pi/B\bar{L}$ is the average level spacing and $\bar{L} = \sum_b L_b/B$ is the mean bond length.)

Universal behavior of the spectral correlation function (agreement with the prediction $R_2^{\rm RMT}$ of RMT) can be expected if the corresponding classical system is chaotic (hyperbolic). What is the equivalent condition on a quantum graph? An answer has been formulated by Tanner [14] (see also [18]) in terms of the classical probability $F_{bd,b'd'} \equiv |S_{bd,b'd'}|^2 = |S(k)_{bd,b'd'}|^2$ to get from (b', d') to (b, d). This "classical propagator" F has one eigenvalue $\lambda_1 = 1$, corresponding to equidistribution in bond space. The dynamics is *mixing* if, for large times, any initial probability distribution converges to this distribution; i.e., $\lim_{n\to\infty}(F^n)_{bd,b'd'} = \frac{1}{2B}$. This condition is met if all other eigenvalues $|\lambda_{2,\ldots,B}| < 1$ lie inside the complex unit circle. However, mixing dynamics alone does not suffice to guarantee universality of a quantum graph [19]. An additional condition proposed by Tanner [14] states that in the limit $B \to \infty$, the spectral gap $\Delta_g = \max_{b\in\{2,\ldots,B\}}(1 - |\lambda_b|)$ is constant or, at least, vanishes slowly enough $\Delta_g \sim B^{-\alpha}$. Building on the so-called diagonal approximation (an approximation that obtains its asymptotics of $R_2(s)$ for large values of p) Tanner conjectured that universal behavior should be expected for values of the gap exponent $0 \le \alpha < 1$.

In the following we show that quantum graphs indeed show RMT spectral correlations (provided the condition $0 \le \alpha < 1/2$ is met somewhat stronger than Tanner's.) We start out from the representation $\rho(k) = \Delta^{-1} - \frac{1}{\pi} \times \frac{d}{dk} \mathrm{Im} \ln \xi(k^+)$, $k^+ \equiv k + i0$ of the density of states in terms of the spectral determinant [10]. Using this formula, it is straightforward to verify that the two-point function assumes the form $R_2(s) = \frac{1}{8\pi^2} \times \frac{d^2}{dj_+ dj_-}|_{j=0} \mathrm{Re}\langle \zeta(j_+, j_-)\rangle_k$, where

$$\zeta(j_+, j_-) \equiv \frac{\xi(k^+ + p_{+\mathbf{f}})}{\xi(k^+ + p_{+\mathbf{b}})} \left(\frac{\xi(k^+ + p_{-\mathbf{f}})}{\xi(k^+ + p_{-\mathbf{b}})}\right)^* \qquad (1)$$

and $p_{\pm\mathbf{b}} = (\pm s/2 - j_\pm)\Delta$, $p_{\pm\mathbf{f}} = (\pm s/2 + j_\pm)\Delta$.

Our analysis is based on the assumption that all bond lengths L_b are rationally independent. It has been shown [13] that under this condition the average over the parameter k is strictly equivalent to an average over B independent phases $e^{i(kL_b/2)} \mapsto e^{i\phi_b}$:

$$\langle \mathcal{F}[T(k)]\rangle_k = \langle \mathcal{F}[T(\phi)]\rangle_\phi, \qquad (2)$$

where $\mathcal{F}$ is a smooth function of the bond-diagonal phase matrix T introduced above.

In order to apply (2) to our present problem, we represent the fraction of determinants in (1) as a Gaussian integral, $\zeta = \int d(\bar{\psi}, \psi) \exp(-S[\bar{\psi}, \psi] + 8\pi i(j_+ - j_-))$, where [20]

$$S[\bar{\psi}, \psi] = \bar{\psi}_+ \begin{bmatrix} 1 & T(k) \\ T(k) & (ST_+)^\dagger \end{bmatrix} \psi_+ + \bar{\psi}_- \begin{bmatrix} 1 & T(k)^\dagger \\ T(k)^\dagger & ST_- \end{bmatrix} \psi_-. \qquad (3)$$

Here, $\psi = \{\psi_{a,s,x,d,b}\}$ is a $16B$-dimensional supervector, where $a = \pm$ distinguishes between the retarded and the advanced sector of the theory (determinants involving S and $S^\dagger$, respectively), $s = \mathbf{f}, \mathbf{b}$ refers to complex commuting and anticommuting components (determinants in the denominator and numerator, respectively), and $x = 1, 2$ to the internal structure of the matrix kernel appearing in (3). Defining $\sigma_i^{\rm bf}$ as the Pauli matrices in superspace, the matrices $T_\pm \equiv T(2p_\pm)$, i.e., $T_\pm$, are diagonal matrices in superspace containing the bond matrices $T(2p_{\pm,\mathbf{b}/\mathbf{f}})$ in the boson-boson/fermion-fermion sector. Using

$$\frac{\xi(k+p)}{\det[ST(2p)]} = \det\begin{bmatrix} 1 & T(k) \\ T(k) & [ST(2p)]^\dagger \end{bmatrix},$$

one verifies that the Gaussian integration over all components of ψ indeed yields the determinant ζ.

As a second step, we subject the phase-averaged ψ functional to a duality transformation known as the color-flavor transformation [15]. In a variant adapted to the present context [21], the transformation states that

$$\langle \exp(-S[\bar{\psi}, \psi])\rangle_\phi = \langle \exp(-S'[\bar{\Psi}, \Psi])\rangle_Z, \qquad (4)$$

where $\langle \cdot \rangle \equiv \int dZ d\tilde{Z} \mathrm{sdet}(1 - Z\tilde{Z})(\cdot)$ and (matrix structure in advanced/retarded space)

$$S'[\bar{\Psi}, \Psi] = \bar{\Psi}_1 \begin{bmatrix} 1 & Z \\ Z^\tau & 1 \end{bmatrix} \Psi_1 + \bar{\Psi}_2 \begin{bmatrix} (ST_+)^\dagger & \tilde{Z}^\tau \\ \tilde{Z} & ST_- \end{bmatrix} \Psi_2. \qquad (5)$$

Referring to [21] for a short discussion of the underlying technicalities, we here briefly explain the notation and the physical meaning of the transformation (4). In (5), $\Psi_{1,2} = \{(\Psi_{1,2})_{a,s,t,b,d}\}$ are $16B$-dimensional independent supervectors, where the index $t = 1, 2$ accounts for the time-reversal symmetry of the model. Presently, all we need to know about the variables Ψ and $\bar{\Psi}$ is that they contain elements of ψ and $\bar{\psi}$ as their components and fulfill $\bar{\Psi}_{1,2} = \Psi_{1,2}^T \tau$. Here, we introduced the fixed supermatrix

$\tau \equiv \sigma_1^{\mathrm{dir}} \otimes \tau_0$, where $\tau_0 \equiv (E_{\mathbf{bb}}\sigma_1^{\mathrm{tr}} - iE_{\mathbf{ff}}\sigma_2^{\mathrm{tr}})$ ($\sigma_i^{\mathrm{tr/dir}}$ are Pauli matrices in the "time-reversal" index t and direction index d; $E_{\mathbf{bb/ff}}$ are projectors on the bosonic/fermionic sector of the theory). The newly introduced integration variables $Z = \mathrm{bdiag}(Z_1, \ldots, Z_B)$ are $8B$-dimensional block supermatrices with eight-dimensional entries $Z_b = \{Z_{b,ss',dd',tt'}\}$. Finally, $Z^\tau \equiv \tau Z^T \tau^{-1}$ is, in a generalized way, transposed to Z, while Z and $\tilde{Z}$ are independent.

What is the physical significance of the transformation (4)? Figure 1 shows a cartoon of the retarded (upper line) and advanced (lower line) wave function dynamics in the system. During propagation, both states pick up random scattering phases T (indicated by vertical dashed lines) and suffer scattering from one bond to the other (S matrix). The rapid succession of these events implies wild fluctuations of the wave function amplitudes. Within the field theoretical context, this translates to uncontrollable fluctuations of the bilinears $\bar{\psi}_{+,s,x,d,b}e^{i\phi_b}\psi_{+,s,x,d,b}$ appearing in the action (3). In contrast, the field Z enters the theory as $\sim\bar{\Psi}_{+,s,t,d,b}Z_{b,ss',tt',dd'}\Psi_{-,s',t',d',b}$, i.e., through structures that *couple* retarded and advanced field amplitudes (the "vertical" ovals in the figure). These amplitudes generally *interfere* to form slowly fluctuating entities (the basic principle behind the formation of universal correlations.) This indicates that the Z integral is comparatively benign and, foreseeably, amenable to stationary phase approximation schemes. To promote this expectation to a quantitative level, we integrate out the Ψ's, thus arriving at the *exact* representation $\langle\zeta\rangle_k = \int dZd\tilde{Z}\exp(-\mathbf{S}[Z,\tilde{Z}])$,

$$\mathbf{S}[Z,\tilde{Z}] = -\mathrm{str}\,\ln(1-\tilde{Z}Z) + \tfrac{1}{2}\mathrm{str}\,\ln(1-Z^\tau Z) + \tfrac{1}{2}\mathrm{str}\,\ln[1-(ST_-)^\dagger\tilde{Z}(ST_+)\tilde{Z}^\tau]. \qquad (6)$$

To better understand this expression, let us consider its quadratic expansion,

$$\mathbf{S}^{(2)}[Z,\tilde{Z}] = \mathrm{str}(\tilde{Z}Z - \tfrac{1}{2}Z^\tau Z - \tfrac{1}{2}T_-^\dagger S^\dagger\tilde{Z}ST_+\tilde{Z}^\tau). \qquad (7)$$

Void of nonlinearities [terms of $\mathcal{O}(Z^4)$], the action $\mathbf{S}^{(2)}$ describes the uninterrupted propagation of two amplitudes along the *same* path in configuration space, i.e., the level of approximation underlying the diagonal approximation in semiclassical periodic-orbit theory.

This connection is made quantitative by noting that the action $\mathbf{S}^{(2)}$ possesses a family of approximately [up to corrections of $\mathcal{O}(B^{-1})$] "massless" configurations, or "zero modes" identified by $\delta_Z\mathbf{S}^{(2)} = \delta_{\tilde{Z}}\mathbf{S}^{(2)} = 0$. Upon substitution of the ansatz $Z_{dd'} = \delta_{dd'}Z_d$ (configurations not diagonal in direction space do not qualify as solutions; see the discussion of deviations below) these equations assume the form

$$\tilde{Z} = Z^\tau, \qquad (\mathbb{1} - \hat{F})Z = 0, \qquad (8)$$

where we have set the parameter matrices $T_\pm = \mathbb{1}$ [22]. Owing to the fact that on a chaotic quantum graph the classical propagator $\hat{F}$ has only one eigenvalue 1, Eq. (8) possesses the unique solution $Z_{b,dd'} = (2B)^{-1/2}\delta_{d,d'}Y$, proportional to the invariant equidistribution. We define $Y^\tau = \tau_0 Y^T \tau_0^{-1}$, where the matrix τ_0 differs from τ by the absence of the (now redundant) matrix σ_1^{dir}. Technically, the relation $\tilde{Y} = Y^\tau$ identifies $(Y,\tilde{Y})$ as generators of the orthosymplectic algebra osp(4|4).

To explore the contribution of these modes to the theory, we set $T_{\pm,b} = \mathbb{1} + iL_b p_\pm$ (provided the bond length fluctuations are not too large, $L_b/\bar{L} = \mathcal{O}(B^0)$, higher orders in the expansion may safely be neglected) do the Gaussian integral over Y and differentiate with respect to the source parameters $j_\pm$. As a result we obtain $R_2^{\mathrm{GOE,diag}} = 1/(\pi^2 s^2)$ which agrees with the $s \gg 1$ asymptotics of the RMT correlation function. This is consistent with the observation that non-Gaussian contributions to the expansion of $\mathbf{S}[Y]$ can no longer be neglected once $s \lesssim 1$.

However, before going beyond the level of the Gaussian approximation, let us briefly consider the role of nonzero mode fluctuations. A glance at Eq. (7) shows that deviations from the first of the two equations in (8) are penalized by a large action $\mathbf{S}^{(2)} = \mathcal{O}(1)$. Upon integration, these modes produce a factor unity to the spectral determinant. Similarly, due to the absence of multiple connectivities and vertex loops, modes that are off diagonal in the direction index d can be integrated out to give a factor unity. To explore the more interesting role played by deviations from the equation $(\mathbb{1} - \hat{F})Z = 0$, let us expand a general configuration $Z_{b,dd'} = \delta_{dd'} \times \sum_{m=1}^{2B} Y_m\chi_{m,bd}$ in the basis of eigenfunctions χ_m of the operator $\hat{F}$. Here, Y_m are four-dimensional supermatrices obeying the symmetry $\tilde{Y}_m = Y_m^\tau$ and the identification $Y_1 \equiv Y$ is understood. For $T_\pm = \mathbb{1}$ the action of the $2B-1 \sim B$ modes $Y_{m>1}$ is given by $\frac{1}{2}\sum_m(1-\lambda_m)\mathrm{str}\tilde{Y}_m Y_m \geq \frac{\Delta_g}{2}\sum_m \mathrm{str}\tilde{Y}_m Y_m$. Gaussian integration over these modes obtains a contribution $\sim\sum_m[\Delta/(1-\lambda_m)]^2 \sim 1/\Delta_g^2 B$ to the spectral function. We conclude that the cumulative contribution of the massive modes can safely be neglected provided $B\Delta_g^2 \overset{B\to\infty}{\longrightarrow} 0$ [23]. (The same holds true for higher order correlation functions, provided the order of the function is smaller than B.)

Going beyond the level of the quadratic approximation, we note that the saddle point equations $\delta_Z\mathbf{S} = \delta_{\tilde{Z}}\mathbf{S} = 0$

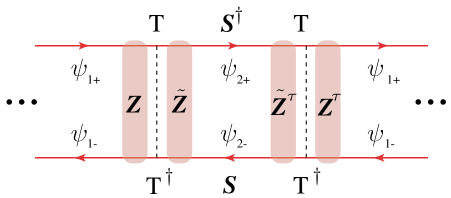

FIG. 1 (color online). On the physical interpretation of the color-flavor transformation. For an explanation, see the text.

of the full action (6) are still solved by the zero mode configurations (8). While deviations from the zero modes continue to be negligible (as long as $B\Delta_g^2 \to \infty$), the action of the latter now reads

$$S[Y] = iB\bar{L}\mathrm{str}[p_-\tilde{Y}Y/(1-\tilde{Y}Y) - p_+Y\tilde{Y}/(1-Y\tilde{Y})],$$

where we have rescaled $Y \to (2B)^{1/2}Y$. To represent this result in a perhaps more widely recognizable form, let us define the 8×8 matrix

$$Q = \begin{pmatrix} 1 & Y \\ \tilde{Y} & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & Y \\ \tilde{Y} & 1 \end{pmatrix}^{-1}.$$

It is then straightforward to verify that the action $S[Y]$ assumes the form of Efetov's [6] action for the GOE correlation function

$$S[Q] = \frac{i\pi}{2}\mathrm{str}(Q\hat{\epsilon}), \tag{9}$$

where $\hat{\epsilon} = -\mathrm{diag}(p_{\mathbf{b}}^+, p_{\mathbf{f}}^+, p_{\mathbf{b}}^-, p_{\mathbf{f}}^-)/\Delta$.

Summarizing, we have proven Tanner's conjecture on universal spectral statistics in large chaotic quantum graphs. Our analysis was based on the assumption of (i) incommensurate bond lengths, (ii) the absence of multiple connectivities, (iii) moderate bond length fluctuations, $L_b/\bar{L} < \mathcal{O}(\mathcal{B})$, and (iv) weak scaling of the spectral gap $\Delta_g \sim B^\alpha, 0 \leq \alpha < 1/2$ of the classical propagator on the graph (which is stronger than Tanner's expectation $0 \leq \alpha < 1$). The conditions (i)–(iv) are met by various families of graphs [11,14,18].

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[22] This simplification is justified because in practice (see below) $T_\pm = \mathbb{1} + \mathcal{O}(B^{-1})$.
[23] In deriving this result, we have neglected the coupling of the modes introduced by $T_\pm$. However, for $L_b/\bar{L} = \mathcal{O}(B^0)$, this coupling leads to corrections to the action of $\mathcal{O}(B^{-1})$ which are negligible.