Contents

Chapter 7

Addition of angular momenta

We know turn to the problem of "adding angular momenta". This is a basic problem that is encountered whenever one has a spin degree of freedom and an orbital angular momentum operator. Then it often makes sense to think of a total angular momentum operator. There are some subtleties involved then, however, which we will take care of in this chapter.

7.1 Spin and the problem of adding angular momenta

7.1.1 Spin operators

We already have a clear understanding of the spin degree of freedom. We know that the Hilbert space $\mathcal{H}$ of a particle is given by

$$\mathcal{H} = L^2(\mathbb{R}) \otimes \mathbb{C}^2. \tag{7.1}$$

This means, of course, if we define the *spin operator* S as

$$S = \frac{\hbar}{2}\sigma, \tag{7.2}$$

with σ being the vector of Pauli matrices, we have that

$$[S, X] = 0, \tag{7.3}$$

$$[S, P] = 0, \tag{7.4}$$

$$[S, L] = 0. \tag{7.5}$$

This is obvious from one perspective, but may take a moment of thought from another. S on the one hand and X, P, and L on the other act on different degrees of freedom, so on different factors in the tensor product. Hence, they clearly commute, by the very definition, and we do not have to compute anything. Also, we have that

$$[S_i, S_j] = i\hbar \sum_k \varepsilon_{i,j,k} S_k, \tag{7.6}$$

for all i, j, so the spin operator satisfies the commutation relation of an angular momentum operator.

7.1.2 Stating the general problem

Let us hence assume that a particle has a spatial degree of freedom as well as one associated with spin. Therefore, we have a L and a S operator. It makes sense to think of the *total angular momentum operator*

$$J = L + S. \tag{7.7}$$

Or, think of two electrons with a spin: Then one would like to again consider the angular momentum operator

$$J = S^{(1)} + S^{(2)}. \tag{7.8}$$

Generally, let us consider the problem of adding

$$J = J^{(1)} + J^{(2)} \tag{7.9}$$

where $J^{(1)}$ and $J^{(2)}$ satisfy the commutation relations of Eq. (7.11). If $J^{(1)}$ and $J^{(2)}$ belong to different degrees of freedom (such as above in Eq. (7.7, 7.8), then they will surely commute

$$[J^{(1)}, J^{(2)}] = 0. \tag{7.10}$$

It also follows that the components of J satisfy

$$[J_i, J_j] = i\hbar \sum_k \varepsilon_{i,j,k} J_k, \tag{7.11}$$

for all i, j, so J is again an angular momentum operator. And all properties that we know of angular momenta apply to J, equally as they have applied to $J^{(1)}$ and $J^{(2)}$ individually.

Let us denote the eigenvectors of $J^{(1)}$ and $J^{(2)}$ by

$$\{|j^{(1)}, m^{(1)}\rangle \, : m^{(1)} = -j^{(1)}, \ldots, j^{(1)}\rangle\}, \tag{7.12}$$

$$\{|j^{(2)}, m^{(2)}\rangle \, : m^{(2)} = -j^{(2)}, \ldots, j^{(2)}\rangle\}, \tag{7.13}$$

respectively. From these we can form the tensor products

$$\{|j^{(1)}, m^{(1)}; j^{(2)}, m^{(2)}\rangle = |j^{(1)}, m^{(1)}\rangle \otimes |j^{(2)}, m^{(2)}\rangle\}. \tag{7.14}$$

These vectors are the eigenvectors of

$$(J^{(1)})^2,\ J_3^{(1)},\ (J^{(2)})^2,\ J_3^{(2)} \tag{7.15}$$

by definition, with eigenvalues

$$\hbar^2 j^{(1)}(j^{(1)}+1),\ \hbar m^{(1)},\ \hbar^2 j^{(2)}(j^{(2)}+1),\ \hbar m^{(2)}, \tag{7.16}$$

by definition: This notation might look a bit clumsy, but this is just what we already know. These vectors are also eigenvalues of the third component J_3 of J, with eigenvalue $\hbar(m^{(1)}+m^{(2)}$. But they are *not* eigenvectors of J^2, since

$$[J^2, J_3^{(1)}] \neq 0,\ [J^2, J_3^{(2)}] \neq 0. \tag{7.17}$$

However, for many applications, one would like to find eigenvectors of J^2 in a similar way as we had known eigenvectors of $(J^{(1)})^2$ and $(J^{(2)})^2$ individually. In yet other words, we want to find the simultaneous eigenvectors of

$$J^2,\ J_3,\ (J^{(1)})^2,\ (J^{(2)})^2. \tag{7.18}$$

We approach this problem by first looking at two spins, then the orbital angular momentum and a spin, and then have a look at the general problem.

7.1.3 Coupling of two spins

We have two spin degrees of freedom $S^{(1)}$ and $S^{(2)}$ and the total angular momentum operator

$$J = S^{(1)} + S^{(2)}. \tag{7.19}$$

The eigenvectors

$$\{|0,0\rangle,\ |0,1\rangle,\ |1,0\rangle,\ |1,1\rangle\} \tag{7.20}$$

form the eigenvectors of $(S^{(1)})^2$, $(S^{(2)})^2$, $S_3^{(1)}$, $S_3^{(2)}$. More specifically,

$$J_3|1,1\rangle = \hbar|1,1\rangle, \tag{7.21}$$
$$J_3|1,0\rangle = 0, \tag{7.22}$$
$$J_3|0,0\rangle = -\hbar|0,0\rangle, \tag{7.23}$$
$$J_3|0,1\rangle = 0. \tag{7.24}$$

What is more, we have that

$$\begin{aligned} J^2 &= (S^{(1)})^2 + (S^{(2)})^2 + 2S^{(1)} \cdot S^{(2)} \\ &= \frac{3}{2}\hbar^2 + 2S_3^{(1)}S_3^{(2)} + S_+S_- + S_-S_+. \end{aligned} \tag{7.25}$$

We also know the following

$$J^2|1,1\rangle = \left(\frac{3}{2}\hbar^2 + 2\left(\frac{\hbar}{2}\right)^2\right)|1,1\rangle = 2\hbar^2|1,1\rangle, \qquad (7.26)$$

$$J^2|0,0\rangle = 2\hbar^2|0,0\rangle. \qquad (7.27)$$

The vectors $|1,1\rangle$ and $|0,0\rangle$ therefore have total spin 1 and a z-component of $\pm\hbar$. The missing eigenvectors for total spin 1 we get by applying J_- to $|1,1\rangle$ and normalize

$$\frac{1}{\hbar\sqrt{2}}J_-|1,1\rangle = \frac{1}{\sqrt{2}}\left(|1,0\rangle + |0,1\rangle\right). \qquad (7.28)$$

In this way, we have found all three eigenvectors with total spin 1. There is another eigenvector, for total spin 0, given by

$$\frac{1}{\sqrt{2}}\left(|1,0\rangle - |0,1\rangle\right). \qquad (7.29)$$

For this we have that

$$J_3\frac{1}{\sqrt{2}}\left(|1,0\rangle - |0,1\rangle\right) = 0, \; J^2\frac{1}{\sqrt{2}}\left(|1,0\rangle - |0,1\rangle\right) = 0. \qquad (7.30)$$

Total angular momentum eigenvectors for two spins: In the notation $|J,m\rangle_J$ of the total spin operator we have that

$$|1,1\rangle_J = |1,1\rangle, \qquad (7.31)$$

$$|1,0\rangle_J = \frac{1}{\sqrt{2}}\left(|1,0\rangle + |0,1\rangle\right), \qquad (7.32)$$

$$|1,-1\rangle_J = |0,0\rangle, \qquad (7.33)$$

$$|0,0\rangle_J = \frac{1}{\sqrt{2}}\left(|1,0\rangle - |0,1\rangle\right). \qquad (7.34)$$

The first three eigenvectors are called *triplet vectors* – spanning the three-dimensional *triplet eigenspace* – for obvious reasons. The last one is the *singlet*. Two of them are product vectors, and the other two are maximally entangled.

7.2 The general problem for two systems

7.2.1 Possible values

Before we turn to the second important special case of considering orbital angular momentum and spin,

$$J = L + S. \qquad (7.35)$$

let us discuss the general case for two subsystems first, and then turn again to this special case. We know that

$$(J^{(i)})^2|j^{(1)},m^{(1)};j^{(2)},m^{(2)}\rangle = \hbar^2 j^{(1)}(j^{(1)}+1)|j^{(1)},m^{(1)};j^{(2)},m^{(2)}\rangle \quad (7.36)$$
$$J_3^{(i)}|j^{(1)},m^{(1)};j^{(2)},m^{(2)}\rangle = \hbar m^{(i)}|j^{(1)},m^{(1)};j^{(2)},m^{(2)}\rangle \quad (7.37)$$

for $i = 1, 2$. For fixed values of $j^{(1)}$ and $j^{(2)}$ these vectors span a $(2j^{(1)}+1)\times(2j^{(2)}+1)$ dimensional space. We call this space $\mathcal{H}(j^{(1)}, j^{(2)})$. Since J^2 commutes with $(J^{(1)})^2$ and $(J^{(2)})^2$, one can find the vectors in $\mathcal{H}(j^{(1)}, j^{(2)})$ as simultanous eigenvectors of

$$(J^{(1)})^2,\ (J^{(2)})^2,\ J^2,\ J_3. \quad (7.38)$$

These vectors are denoted by

$$\{|j^{(1)},j^{(2)};j;m\rangle_J\}. \quad (7.39)$$

They satisfy by definition

$$(J^{(i)})^2|j^{(1)},j^{(2)};j;m\rangle_J = \hbar^2 j^{(i)}(j^{(i)}+1)|j^{(1)},j^{(2)};j;m\rangle_J, \quad (7.40)$$
$$J^2|j^{(1)},j^{(2)};j;m\rangle_J = \hbar^2 j(j+1)|j^{(1)},j^{(2)};j;m\rangle_J, \quad (7.41)$$
$$J_3|j^{(1)},j^{(2)};j;m\rangle_J = \hbar m|j^{(1)},j^{(2)};j;m\rangle_J. \quad (7.42)$$

The remaining problem is to determine, given $j^{(1)}$ and $j^{(2)}$ the possible values that j and m can take. In order to approach this problem, consider

$$J_3|j^{(1)},m^{(1)};j^{(2)},m^{(2)}\rangle = \hbar(m^{(1)}+m^{(2)})|j^{(1)},m^{(1)};j^{(2)},m^{(2)}\rangle. \quad (7.43)$$

For a given m, let us denote with $n(m)$ the number of orthogonal state vectors $|j^{(1)},m^{(1)};j^{(2)},m^{(2)}\rangle$ such that

$$m^{(1)}+m^{(2)} = m. \quad (7.44)$$

By direct inspection and a moment of thought (and possibly a figure), one finds that

$$n(m) = \begin{cases} 0, & \text{if } |m| > j^{(1)}+j^{(2)}, \\ j^{(1)}+j^{(2)}+1-|m|, & \text{if } j^{(1)}+j^{(2)} \geq |m| \geq |j^{(1)}-j^{(2)}|, \\ \min(2j^{(1)}+1, 2j^{(2)}+1), & \text{if } |j^{(1)}-j^{(2)}| \geq |m| \geq 0. \end{cases} \quad (7.45)$$

We can now use a trick that we have used several times before: If

$$|j^{(1)},j^{(2)};j;m\rangle_J \in \mathcal{H}(j^{(1)},j^{(2)}), \quad (7.46)$$

then also all vectors

$$|j^{(1)},j^{(2)};j;m'\rangle_J \in \mathcal{H}(j^{(1)},j^{(2)}), \quad (7.47)$$

with

$$m' = -j, -j+1 \ldots, j-1, j. \tag{7.48}$$

We merely have to apply the ladder operators $J_\pm^{(1)} + J_\pm^{(2)}$ repeatedly and to take into account that the space is stable under the action of these ladder operators. If then $N(j)$ denotes the degeneracy of the total angular momentum J, we have

$$n(m) = \sum_{j \geq |m|} N(j). \tag{7.49}$$

From this it follows that

$$N(j) = n(j) - n(j+1), \tag{7.50}$$

and employing Eq. (7.45), we get

$$N(j) = \begin{cases} 0, \text{ if } j > j^{(1)} + j^{(2)},\ j < |j^{(1)} - j^{(2)}|, \\ 1, \text{ if } |j^{(1)} - j^{(2)}| \leq j \leq j^{(1)} + j^{(2)}. \end{cases} \tag{7.51}$$

To summarize, for given $j^{(1)}$ and $j^{(2)}$, the possible values of j are

$$|j^{(1)} - j^{(2)}|, |j^{(1)} - j^{(2)}| + 1, \ldots, j^{(1)} + j^{(2)}. \tag{7.52}$$

For each value of j there is a series of eigenvectors

$$\{|j^{(1)}, j^{(2)}; j; m\rangle_J : m = -j, \ldots, j\}. \tag{7.53}$$

7.2.2 Glebsch-Gordan coefficients

The coefficients relating the two bases are usually referred to as Glebsch-Gordan coefficients:

Glebsch-Gordon coefficients: The coefficients in

$$|j^{(1)}, j^{(2)}; j; m\rangle_J = \sum_{m^{(1)}, m^{(2)}, m = m^{(1)} + m^{(2)}} C(j^{(1)}, j^{(2)}, j, m^{(1)}, m^{(2)}, m) |j^{(1)}, m^{(1)}; j^{(2)}, m^{(2)}\rangle \tag{7.54}$$

are called *Glebsch-Gordon coefficients*.

7.2.3 An optional remark on group theory

For those familiar with the language of group theory, this can be formulated as follows: For given irreducible representations $\mathcal{D}^{(j^{(1)})}$ and $\mathcal{D}^{(j^{(2)})}$ of the rotation group, with basis vectors $\{|j^{(1)}, m^{(1)}\rangle\}$ and $\{|j^{(2)}, m^{(2)}\rangle\}$, the vectors

$$\{|j^{(1)}, m^{(1)}; j^{(2)}, m^{(2)}\rangle\} \tag{7.55}$$

give rise to a representation of the group which is in general reducible. This can be decomposed into the direct sum form

$$\mathcal{D}^{(j^{(1)})} \otimes \mathcal{D}^{(j^{(2)})} = \mathcal{D}^{|j^{(1)}-j^{(2)}|} \oplus \mathcal{D}^{|j^{(1)}-j^{(2)}|+1} \oplus \cdots \oplus \mathcal{D}^{j^{(1)}+j^{(2)}}. \tag{7.56}$$

What is this supposed to mean? What is a group in the first place?

A group is a set, G, together with an operation $\cdot$ combining to elements of G, written as $g = a \cdot b$, such that the following properties are satisfied:

- (Closure) For all $a, b \in G$, $a \cdot b \in G$.
- (Associativity) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in G$.
- (Identity element) There exists an element $e \in G$ such that

  $$a \cdot e = e \cdot a = a \tag{7.57}$$

  for all $a \in G$.
- (Inverse element) For each $a \in G$, there exists an inverse element $b \in G$ such that

  $$a \cdot b = b \cdot a = e. \tag{7.58}$$

Permutations form a group. So do rotations. In fact, the group referred to above is the rotation group. A representation is a group homomorphism

$$V : G \to GL(n, \mathbb{R}) \tag{7.59}$$

from the group to the general linear group. Roughly speaking, an irreducible representation is a "smallest possible" representation, one that cannot be broken down to smaller components. A reducible representation can be decomposed into a direct sum of blocks, just as we have seen a minute ago. To dive into the topic of representation theory is beyond the scope of this course (but it is useful, so take this as an invitation).

7.2.4 Adding orbital angular momentum and a spin

Now, in case of

$$J = L + S, \tag{7.60}$$

the state vectors

$$\{|l, m_l\rangle|0\rangle,\ |l, m_l\rangle|1\rangle\} \tag{7.61}$$

are eigenvectors of L^2, S^2, L_3 and S_3, but not of J^2. We can follow the above machinery, though, and find the values

$$j = l + \frac{1}{2}, l - \frac{1}{2} \tag{7.62}$$

for j.