Freie Universität Berlin
Analytische Mechanik
Sommer-Semester 2022

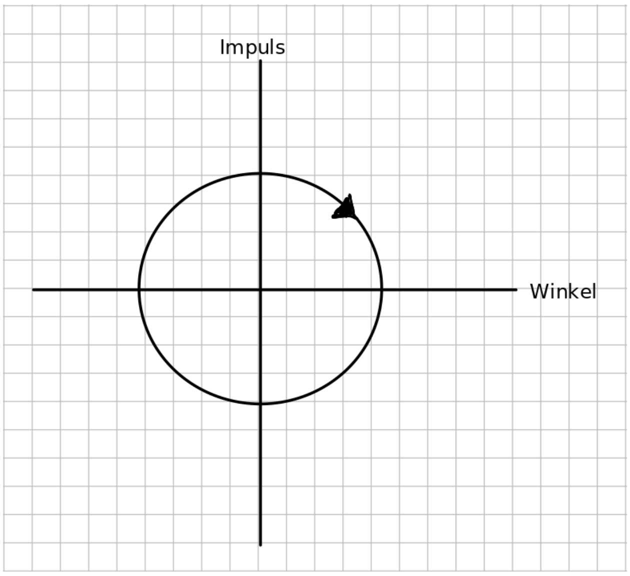
Klausur
22.07.2022 um 10:00

Aufgabe 1: Kurze Fragen — kurze Anworten $[1 + 1 + 2 + 1 + 2 = 7$ Punkte$]$

Die vollständige Punktzahl kann mit kurzen Antworten (z.B. 2-3 Sätze) und ohne Rechnungen erreicht werden.

(a) Betrachten Sie ein Pendel, das unter Einfluss der Gravitationskraft in der (x, y)-Ebene schwingt. Skizzieren Sie die Trajektorie im Phasenraum! *[1 pt]*

Lösung:



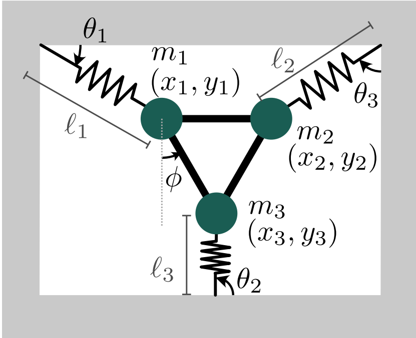
(b) Betrachten Sie einen Stern der Masse M am Punkt $(0, 0, 0)$ der mit einer Gravitationskraft auf einen Planeten der Masse m, mit $m \ll M$, wirkt. Nennen Sie mögliche Formen der Trajektorien, auf denen sich der Planet bewegen kann. *[1 pt]*

Lösung:

- Elliptical orbit;
- Parabolic orbit;
- Hyperbolic orbit.

(c) Betrachten Sie das System dreier gekoppelter Massen in der Darstellung rechts. Die Massen sind untereinander mit starren Stäben verbunden und können sich in der Ebene bewegen. Zusätzlich ist jede Masse einzeln mittels einer Feder an der Wand befestigt. *[2 pt]*

Wie viele Freiheitsgrade hat das System (mit kurzer Begründung)? Geben Sie geeignete generalisierte Koordinaten an.



Lösung: Das System hat drei Freiheitsgrade. Ohne Zwangsbedingungen hätten 3 Massen in der Ebene $3 \times 2 = 6$ Freiheitsgrade, 3 Stangen bringen 3 skalare Zwangsbedingung (Die paarweise Abstände zwischen jede Massen sind fixiert). Federn bringen keine Zwangsbedingungen.

Eine mögliche Wahl von verallgemeinerte Koordinaten ist (x_1, y_1, ϕ). Mögliche Alternative:

- (x_i, y_i, ϕ), for any given i;
- (x_i, y_i, θ_j) for any given $i \neq j$;
- (x_i, y_i, x_j), for any $i \neq j$;
- $(\theta_1, \theta_2, \theta_3)$;
- (x_1, y_1, ℓ_2) — borderline wrong, because there are two possible positions for the same values of the coordinates, would only work for small perturbations around a point;
- etc.

Falsch wäre aber z.B.:

- (x_1, x_2, x_3) — then y_j, ℓ_j undetermined;
- (ℓ_1, x_1, y_1) — ℓ_1 is already determined by (x_1, y_1) and then the other masses' positions are undetermined;
- etc.

(d) Was ist eine kanonische Transformation? *[1 pt]*

Lösung: Canonical transformations are transformations of the degrees of freedom (q_j, p_j) and of the Hamiltonian H with the property that that preserve the Poisson bracket. They are an alternative description of the system with the same physics (same physical solutions).

Alternatively: Canonical transformations are transformations of the degrees of freedom (q_j, p_j) such that the form of Hamilton's equations of motion are preserved (note that this definition can differ subtly from the previous one).

Alternatively: Canonical transformations are transformations of the degrees of freedom (q_j, p_j) such that the Jacobian of the transformation is a symplectic matrix.

(e) Welche Erhaltungsgrößen sind mit welchen Symmetrien der Lagrangefunktion verknüpft? Ergänzen Sie die folgende Tabelle. *[2 pt]*

Symmetrie von L	*Erhaltungsgrösse(n)*
? (A)	Hamiltonfunktion
Invarianz unter Translationen in der x-Richtung	? (B)
? (C)	y-Komponente der Drehimpuls
Invarianz unter beliebige Rotationen in 3D	? (D)

Lösung:

(A) The transformation $t \to t + \text{const}$, i.e., translations in time. Alternatively: the transformation of the generalized coordinates that amounts to a time evolution with the current equations of motion.

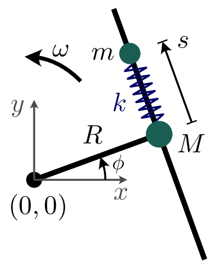
(B) The x-component of the momentum, i.e., $p_x = m\dot{x}$.

(C) Invariance under any spatial rotation about the y axis.

(D) The angular momentum vector.

Aufgabe 2: Rotierendes T-Stück mit Feder [$1 + 2 + 1 + 2 + 2 + 2 + 2 + 2 = 14$ Punkte]

Eine masselose Stange der Länge R ist an einem Ende an einer Aufhängung am Umsprung $(0, 0)$ der x-y Ebene befestigt. Am anderen Ende ist eine zweite sehr lange masselose Stange befestigt, die im rechten Winkel zur ersten Stange montiert ist. Beide Stangen bewegen sich nur in der x-y Ebene. Die erste Stange wird um $(0, 0)$ mit konstanter Winkelgeschwindigkeit ω getrieben. Am Verbindungspunkt der beiden Stangen sei eine Masse M fixiert. Eine zweite Masse m kann sich entlang der langen Stange frei bewegen, ist jedoch mittels einer Feder mit Federkonstante k und Ruhelänge ℓ_0 mit M verbunden. Es wirkt *keine* Gravitationskraft.



[Correction: It was announced during the exam that the indication $\ell_0 = 0$ was missing.]

(a) Wie viele Freiheitsgräde hat das System? Begründen Sie! Wählen Sie geeignete verallgemeinerte Koordinaten. *[1 pt]*

Lösung: There's only a single degree of freedom. The mass M is not free at all; it is constrained to rotate at radius R and at angular velocity ω (no degrees of freedom); the mass m is constrained to move along the bar (one degree of freedom). A good choice of generalized coordinate is s.

(b) Zeigen Sie, dass die kinetische Energie der Masse m wie folgt geschrieben werden kann: *[2 pt]*

$$T_m = \frac{1}{2}m\left[\left(R\omega + \dot{s}\right)^2 + \omega^2 s^2\right] \; . \tag{1}$$

(c) Stellen Sie die Lagrangefunktion des Systems auf! *[1 pt]*

Lösung: The kinetic energy of the mass M is

$$T_M = \frac{1}{2}MR^2\omega^2 \; . \tag{s.1}$$

It's a constant (it doesn't depend on s or $\dot{s}$), and so it won't influence the equations of motion of the Lagrangian. We'll omit it in the Lagrangian, although it is perfectly correct to include it as well.

To compute the kinetic energy of the mass m, we first write

$$\boldsymbol{r}_m = \begin{pmatrix} R\cos\phi + s\cos\left(\phi + \frac{\pi}{2}\right) \\ R\sin\phi + s\sin\left(\phi + \frac{\pi}{2}\right) \end{pmatrix} = \begin{pmatrix} R\cos\phi - s\sin\phi \\ R\sin\phi + s\cos\phi \end{pmatrix} = \begin{pmatrix} R\cos(\omega t) - s\sin(\omega t) \\ R\sin(\omega t) + s\cos(\omega t) \end{pmatrix} \; , \tag{s.2}$$

with $\phi = \omega t$, and

$$\dot{\boldsymbol{r}}_m = \begin{pmatrix} -R\omega\sin(\omega t) - \dot{s}\sin(\omega t) - s\omega\cos(\omega t) \\ R\omega\cos(\omega t) + \dot{s}\cos(\omega t) - s\omega\sin(\omega t) \end{pmatrix} = \left(R\omega + \dot{s}\right)\begin{pmatrix} -\sin(\omega t) \\ \cos(\omega t) \end{pmatrix} + s\omega\begin{pmatrix} \cos(\omega t) \\ \sin(\omega t) \end{pmatrix} \; . \tag{s.3}$$

The two vectors in the last expression are manifestly orthogonal, so we can directly compute

$$\dot{\boldsymbol{r}}_m^2 = \left(R\omega + \dot{s}\right)^2 + \omega^2 s^2 \; . \tag{s.4}$$

The kinetic energy of the mass m is therefore

$$T_m = \frac{1}{2}m\left[\left(R\omega + \dot{s}\right)^2 + \omega^2 s^2\right] \; . \tag{s.5}$$

expression of $\boldsymbol{r}_m$ and 1pt for correct expression of $\dot{\boldsymbol{r}}_m$.]

The potential term is simply given by the spring. It is

$$V = \frac{1}{2}ks^2 \; . \tag{s.6}$$

The Lagrangian is therefore (different choices up to a constant)

$$L = T_m - V = \frac{1}{2}m\left[\left(R\omega + \dot{s}\right)^2 + \omega^2 s^2\right] - \frac{1}{2}ks^2 \; . \tag{s.7}$$

(d) Zeigen Sie, dass die Bewegungsgleichung für s wie folgt geschrieben werden kann: *[2 pt]*

$$m\ddot{s} = \left(\omega^2 m - k\right) s \ . \tag{2}$$

Lösung: We compute on one hand the partial derivative

$$\frac{\partial L}{\partial s} = \left(m\omega^2 - k\right) s \ . \tag{s.8}$$

On the other hand

$$\frac{\partial L}{\partial \dot{s}} = m\left(R\omega + \dot{s}\right) \ , \tag{s.9}$$

leading to

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{s}} = m\ddot{s} \ . \tag{s.10}$$

The corresponding Euler-Lagrange equation gives the following equation of motion:

$$m\ddot{s} = \left(m\omega^2 - k\right) s \ . \tag{s.11}$$

(e) Finden Sie alle Gleichgewichtspunkte von s. *[2 pt]*

Lösung: To identify equilibrium points we set $\ddot{s} = 0$ in (2). We obtain

$$s\left(\omega^2 m - k\right) = 0 \ . \tag{s.12}$$

We identify two situations:

- if $\omega^2 m - k \neq 0$: The only equilibrium point is $s = 0$
- if $\omega^2 m - k = 0$: All values $s \in \mathbb{R}$ are equilibrium points.

equilibrium points, 0.5pt each.]

(f) Bestimmen Sie für jeden Gleichgewichtspunkt von s ob dieser stabil ist. Falls ja, berechnen Sie die Oszillationsfrequenz kleiner Schwingungen um den Gleichgewichtspunkt! *[2 pt]*

Lösung: First we treat the case $\omega^2 m - k = 0$. In this case the equations of motion are simply $m\ddot{s} = 0$ (for all s). This is the equation of a free particle in 1 dimension. This happens because the spring cancels out exactly the centrifugal force at all displacements s. In this case, the solution is not a stable oscillation (there's no force that will bring back the mass m towards its initial point).

Now we treat the case $\omega^2 m - k \neq 0$. Now (2) is simply the equation of a harmonic oscillator or of an exponentially divergent solution, depending on the sign of $\omega^2 m - k$. (The equation of motion (2) should normally be expanded to first order in s about $s \approx 0$, but that equation is already linear so no expansion is necessary.)

If $\omega^2 m - k > 0$, then (2) describes an exponentially diverging solution, given that the coefficient of s is positive. This is an unstable point.

If $\omega^2 m - k < 0$, we find that (2) is the equation of a harmonic oscillator. It describes a stable oscillation with angular frequency

$$\Omega = \sqrt{\frac{k}{m} - \omega^2} \ . \tag{s.13}$$

(g) Berechnen Sie die verallgemeinerten Impulse und stellen Sie die Hamiltonfunktion auf. Ist die Hamiltonfunktion erhalten? Begründen Sie! *[2 pt]*

Lösung: The generalized momentum $p \equiv p_s$ associated with the generalized coordinate s is

$$p = \frac{\partial L}{\partial \dot{s}} = m\big(R\omega + \dot{s}\big) \ . \tag{s.14}$$

We can invert the above relation to obtain

$$\dot{s} = \frac{p}{m} - R\omega \ . \tag{s.15}$$

The Hamiltonian is then given by

$$\begin{aligned} H &= \dot{s}p - L \\ &= \frac{p^2}{m} - R\omega p - \left(\frac{1}{2}m\left[\left(\frac{p}{m}\right)^2 + \omega^2 s^2\right] - \frac{1}{2}ks^2\right) \\ &= \frac{p^2}{2m} - R\omega p + \frac{1}{2}\big(-m\omega^2 + k\big)\, s^2 \ . \end{aligned} \tag{s.16}$$

The Hamiltonian has no explicit dependence on time t, i.e., $\partial H/\partial t = 0$, meaning that H is conserved.

(h) Berechnen Sie die Gesamtenergie des Systems. Wie ändert sich diese mit der Zeit? *[2 pt]*

Lösung: The total energy is given by

$$T+V = T_M + T_m + V = \frac{1}{2}M\omega^2R^2 + \frac{1}{2}m\Big[\Big(\frac{p}{m}\Big)^2 + \omega^2 s^2\Big] + \frac{1}{2}ks^2$$
$$= \frac{1}{2}M\omega^2R^2 + \frac{p^2}{2m} + \frac{1}{2}\big(m\omega^2 + k\big)\, s^2 \ . \tag{s.17}$$

We do not expect the total energy to be conserved, since the system is driven; i.e., there is an external system that furnishes work.

Method #1 (Poisson brackets): The time derivative of $T+V$ can be computed via Poisson brackets, noting that $\partial T/\partial t = \partial V/\partial t = 0$:

$$\begin{aligned}\frac{d}{dt}\big(T+V\big) &= \big\{T+V\,,\,H\big\}\\ &= \Big\{\frac{p^2}{2m} + \frac{1}{2}\big(m\omega^2+k\big)\,s^2\ ,\ \frac{p^2}{2m} - R\omega p + \frac{1}{2}\big(-m\omega^2+k^2\big)\,s^2\Big\}\\ &= \frac{1}{2m}\,\frac{1}{2}\big(-m\omega^2+k^2\big)\,\big\{p^2,s^2\big\} + \frac{1}{2}\big(m\omega^2+k\big)\frac{1}{2m}\,\big\{s^2,p^2\big\}\\ &\quad + \frac{1}{2}\big(m\omega^2+k\big)(-R\omega)\,\big\{s^2,p\big\}\ .\end{aligned} \tag{s.18}$$

We recall the canonical Poisson brackets $\{s,p\} = 1$; then $\{s^2,p\} = s\{s,p\} + \{s,p\}s = 2s$ and $\{s^2,p^2\} = p\{s^2,p\} + \{s^2,p\}p = 4sp$. Finally we find

$$(\mathrm{s}.18) = \omega^2 sp + \omega^2 sp - R\omega\big(m\omega^2+k\big)\,s = 2\omega^2 sp - R\omega s\big(m\omega^2+k\big)\ . \tag{s.19}$$

The total energy is clearly not a conserved quantity. (The only solution that happens to conserve the total energy is $s(t)=0$, $p(t)=0$.)

Method #2 (Equations of motion): We can also compute the time derivative of $T+V$ directly using the equations of motion. First, from the Hamiltonian equations of motion we find

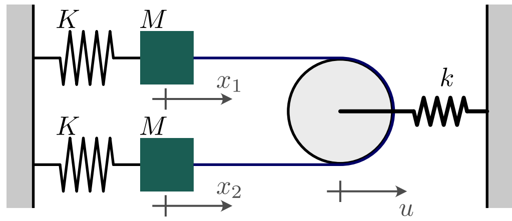
$$\frac{d}{dt}s = \frac{\partial H}{\partial p} = \frac{p}{m} - R\omega\ ; \qquad\qquad \frac{d}{dt}p = -\frac{\partial H}{\partial s} = \big(m\omega^2 - k\big)\,s\ . \tag{s.20}$$

Then we compute

$$\begin{aligned}\frac{d}{dt}\big(T+V\big) &= \frac{p}{m}\frac{dp}{dt} + \big(m\omega^2+k\big)\,s\,\frac{ds}{dt}\\ &= \frac{p}{m}\big(m\omega^2-k\big)\,s + \big(m\omega^2+k\big)\Big(\frac{p}{m} - R\omega\Big)\,s\\ &= 2\omega^2 sp - R\omega s\big(m\omega^2+k\big)\ .\end{aligned} \tag{s.21}$$

Aufgabe 3: Über Umlenkrolle gekoppelte Massen [$4+4=8$ Punkte]

Betrachten Sie das System in der rechten Abbildung. Zwei Massen der Masse M und die (masselose) Umlenkrolle können sich nur horizontal bewegen und sind über die abgebildeten Federn (mit Federkonstanten K bzw. k) an den Wänden befestigt. Die Federn können ebenfalls als masselos angenommen werden. Nehmen Sie außerdem an, dass die Wände ausreichend weit voneinander entfernt sind, sodass sich die Massen und die Umlenkrolle nicht berühren können.



(a) Wählen Sie geeignete verallgemeinerten Koordinaten, stellen Sie die Lagrangefunktion auf und leiten sie die Hamiltonfunktion her! *[4 pt]*

Lösung: A natural choice of generalized coordinates are x_1, x_2. Then $u = (x_1 + x_2)/2$. The kinetic energy is

$$T = \frac{1}{2} M \left(\dot{x}_1^2 + \dot{x}_2^2\right) , \tag{s.22}$$

and the potential energy is

$$V = \frac{1}{2} K x_1^2 + \frac{1}{2} K x_2^2 + \frac{1}{2} k u^2 = \frac{1}{2} K \left(x_1^2 + x_2^2\right) + \frac{1}{8} k \left(x_1 + x_2\right)^2 . \tag{s.23}$$

We obtain the Lagrangian

$$L(x_1, x_2, \dot{x}_1, \dot{x}_2, t) = \frac{1}{2} M \left(\dot{x}_1^2 + \dot{x}_2^2\right) - \frac{1}{2} K \left(x_1^2 + x_2^2\right) - \frac{1}{8} k \left(x_1 + x_2\right)^2 . \tag{s.24}$$

The generalized momenta are

$$p_1 = \frac{\partial L}{\partial \dot{x}_1} = M \dot{x}_1 ; \qquad p_2 = \frac{\partial L}{\partial \dot{x}_2} = M \dot{x}_2 . \tag{s.25}$$

Because the system is two one-dimensional particles in Cartesian coordinates with some potential, we can immediately write the corresponding Hamiltonian as

$$H = \frac{p_1^2}{2M} + \frac{p_2^2}{2M} + V = \frac{p_1^2}{2M} + \frac{p_2^2}{2M} + \frac{1}{2} K \left(x_1^2 + x_2^2\right) + \frac{1}{8} k \left(x_1 + x_2\right)^2 . \tag{s.26}$$

(b) Nehmen Sie an, dass das System zum Zeitpunkt $t = 0$ in folgendem Zustand initialisiert wird: *[4 pt]*

$$x_1(t{=}0) = -A \qquad x_2(t{=}0) = -A \qquad u(t{=}0) = -A \tag{3a}$$

$$\dot{x}_1(t{=}0) = 0 \qquad \dot{x}_2(t{=}0) = 0 \qquad \dot{u}(t{=}0) = 0 , \tag{3b}$$

wobei $A > 0$. Für $t > 0$ lässt man das System sich nun frei bewegen. Leiten Sie die Trajektorie $u(t)$ der Rolle als Funktion der Zeit her!

Lösung: **Method #1: Rewrite the Hamiltonian in terms of u.** Let's transform our coordinates from x_1, x_2 to u, v, where $u = (x_1 + x_2)/2$ and $v = (x_1 - x_2)/2$. Then $x_1 = u + v$, $x_2 = u - v$,

$\dot{x}_1 = \dot{u} + \dot{v}$, $\dot{x}_2 = \dot{u} - \dot{v}$, and

$$p_u = \frac{\partial L}{\partial \dot{u}} = \frac{\partial L}{\partial \dot{x}_1}\frac{\dot{x}_1}{\dot{u}} + \frac{\partial L}{\partial \dot{x}_2}\frac{\dot{x}_2}{\dot{u}} = M\dot{x}_1 + M\dot{x}_2 = p_1 + p_2 \ ; \tag{s.27}$$

$$p_v = \frac{\partial L}{\partial \dot{v}} = \frac{\partial L}{\partial \dot{x}_1}\frac{\dot{x}_1}{\dot{v}} + \frac{\partial L}{\partial \dot{x}_2}\frac{\dot{x}_2}{\dot{v}} = M\dot{x}_1 - M\dot{x}_2 = p_1 - p_2 \ . \tag{s.28}$$

Then $p_1^2 + p_2^2 = \frac{1}{2}(p_u^2 + p_v^2)$, and $x_1^2 + x_2^2 = 2(u^2 + v^2)$, and

$$H = \frac{p_u^2}{4M} + \frac{p_v^2}{4M} + \frac{1}{2}K\,2\big(u^2 + v^2\big) + \frac{1}{2}ku^2 = H_u + H_v \ , \tag{s.29}$$

with $H_u = p_u^2/(4M) + (K + \frac{1}{2}k)u^2$ and $H_v = p_v^2/(4M) + Kv^2$ two independent harmonic oscillators, they're normal modes. From Hamilton's equations of motion we find

$$\frac{du}{dt} = \frac{\partial H}{\partial p_u} = \frac{p_u}{2M} \ ; \qquad\qquad \frac{dp_u}{dt} = -\frac{\partial H}{\partial u} = -(2K + k)\,u \ , \tag{s.30}$$

so we find

$$\ddot{u}(t) = -\frac{2K + k}{2M}\,u \quad \rightarrow \quad u(t) = B\cos(\omega t + \phi) \ , \tag{s.31}$$

where $\omega = \sqrt{\frac{K+k/2}{M}}$ and where B, ϕ are to be determined by initial conditions. Setting $u(t{=}0) = -A$, $\dot{u}(t{=}0) = 0$ we find $\phi = 0$, $B = -A$ (or $B = A$ and $\phi = \pi$), and therefore

$$u(t) = -A\cos(\omega t) \ . \tag{s.32}$$

Method #2: Calculate normal modes & solutions via Euler-Lagrange. The Euler-Lagrange equations give us

$$\begin{aligned} 0 &= \frac{\partial L}{\partial x_1} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} = -Kx_1 - \frac{1}{4}k(x_1 + x_2) - M\ddot{x}_1 \ ; \\ 0 &= \frac{\partial L}{\partial x_2} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_2} = -Kx_2 - \frac{1}{4}k(x_1 + x_2) - M\ddot{x}_2 \ , \end{aligned} \tag{s.33}$$

giving us

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\frac{1}{M}\begin{pmatrix} K + k/4 & k/4 \\ k/4 & K + k/4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \ . \tag{s.34}$$

The matrix is $K\mathbb{1} + (k/4)\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, which has normalized eigenvectors $(1, \pm 1)/\sqrt{2}$ associated with eigenvalues $K + k/4 \pm k/4$. The normal modes are thus

$$u_\pm = \big(x_1 \pm x_2\big)/\sqrt{2} \ , \tag{s.35}$$

with $u \equiv u_+/\sqrt{2}$. Therefore the quantity u we are required to analyze is a normal mode, and we find

$$\ddot{u} = -\frac{K + k/2}{M}\,u \quad \rightarrow \quad u(t) = B\cos(\omega t + \phi) \ , \tag{s.36}$$

where $\omega = \sqrt{\frac{K+k/2}{M}}$ and where B, ϕ are to be determined by initial conditions. Setting $u(t{=}0) = -A$,

$\dot{u}(t{=}0) = 0$ we find $\phi = 0$, $B = -A$ (or $B = A$ and $\phi = \pi$), and therefore

$$u(t) = -A\cos(\omega t) \ . \tag{s.37}$$

Method #3: Using symmetry and Newton's second law. First, we notice that the problem as well as the initial conditions are symmetric if we flip the setup upside down, so it must hold that $x_1(t) = x_2(t) = u(t)$ during the entire course of the evolution for those given initial conditions. The forces that act on one of the masses, say mass number 1, are $-Kx_1$ from the left spring and half the force from the pulley's spring, i.e., $-k/2u = -k/2x_1$, so that the total force is

$$F_{\mathrm{tot},1} = -Kx_1 - \frac{k}{2}\,x_1 \ . \tag{s.38}$$

Newton's law on that mass gives us

$$M\ddot{x}_1 = -(K + k/2)\,x_1 \ , \tag{s.39}$$

which is a simple harmonic oscillator which we can solve as above (see other methods) as $x_1(t) = -A\cos(\omega t)$ with $\omega = \sqrt{(K + k/2)/M}$. We already saw that $u(t) = x_1(t)$ because of the symmetry of the problem so we have our solution, $u(t) = -A\cos(\omega t)$.

Other methods are also possible, e.g., to rewrite L in terms of u, v or $u_\pm$.