Freie Universität Berlin
Tutorials for Advanced Quantum Mechanics
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Warm Up Tutorial

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This sheet is intended as a warm-up, and will not be handed in for grades. The solutions will be discussed in the first week of tutorials, however it would be best to attempt to answer as many questions as possible before the tutorial.

1. **Basis Changes**
   Given a finite dimensional vector space V, Suppose A' and A'' are matrix representations of an operator $A : V \to V$, with respect to two different orthonormal bases $\{|v_i\rangle\}$ and $\{|w_i\rangle\}$ for V - i.e. the matrix elements of A' and A'' are $A'_{ij} = \langle v_i|A|v_j\rangle$ and $A''_{ij} = \langle w_i|A|w_j\rangle$ respectively. Characterize the relationship between A' and A''.

2. **Hermitian Operators**
   Consider Hermitian operators A and B:
   (a) Prove that $[A, B] = 0$ if and only if there exists an orthonormal basis such that A and B are both diagonal with respect to that basis.
   (b) Show that $i[A, B]$ is Hermitian.
   (c) Suppose $[A, B] = 0$, show that $e^{A+B} = e^A e^B$.
   (d) What are the conditions on A and B for AB to be Hermitian?
   (e) Prove that two eigenvectors of a Hermitian operator with different eigenvalues are necessarily orthogonal.

3. **Tensor Product**
   (a) Let $|\psi\rangle = (1/\sqrt{2})(|0\rangle + |1\rangle)$. Write out $|\psi\rangle^{\otimes 2}$ and $|\psi\rangle^{\otimes 3}$ explicitly, both in terms of tensor products like $|0\rangle|1\rangle$ and via the Kronecker product.
   (b) Show that the transpose, complex conjugation and adjoint operations distribute over the tensor product, i.e. show that for operators A and B acting on a finite dimensional vector space V one has that

$$(A \otimes B)^* = A^* \otimes B^*, \quad (A \otimes B)^T = A^T \otimes B^T, \quad (A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$$

4. **Commutation Relations**
   Consider two linear operators A and B. Suppose that $[A, B] = 0$, $\{A, B\} = 0$ and A is invertible. Show that B must be 0.

5. **Pauli Matrices and Measurements**
Let us define the Pauli matrices as follows,

$$\begin{aligned}
\sigma_1 = \sigma_x = X &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
\sigma_2 = \sigma_y = Y &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} && (1) \\
\sigma_3 = \sigma_z = Z &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} && (2)
\end{aligned}$$

and let $\boldsymbol{v}$ be any real three-dimensional unit vector and θ a real number.

(a) Prove that

$$e^{i\theta \boldsymbol{v}\cdot\boldsymbol{\sigma}} = \cos(\theta)I + i\sin(\theta)\boldsymbol{v}\cdot\boldsymbol{\sigma}$$

where $\boldsymbol{v}\cdot\boldsymbol{\sigma} = \sum_{i=1}^{3} v_i\sigma_i$.

(b) Calculate the probability of obtaining the result $+1$ for a measurement of $\boldsymbol{v}\cdot\boldsymbol{\sigma}$, given that the state prior to the measurement is $|0\rangle$. What is the state of the system after the measurement if $+1$ is obtained?

6. **Mixed States**
let ρ be a density operator. Show that $\mathrm{Trace}(\rho^2) \leq 1$, with equality if and only if ρ is a pure state.

7. **Position and Momentum**
Show that the position operator $\hat{x} = x$ and the Hamiltonian operator

$$\hat{H} = -(\hbar^2/2m)d^2/dx^2 + V(X)$$

are Hermitian.