Freie Universität Berlin
Advanced Quantum Mechanics
Wintersemester 2018/19

Exercise sheet 1

Due: 26.10.2018 10:15 J. Eisert

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1. **Wave function basics** $[2+2+2+2=8 \text{ Points}]$
A particle of mass m has as a wave function given by:

$$\Psi(x,t) = Ae^{-a(mx^2/\hbar + it)}, \tag{1}$$

where the constants A and a are positive and real.

a) Find A.

b) Find the potential energy function $V(x)$ for which $\Psi(x,t)$ satisfies the Schrödinger equation.

c) Calculate the expectation values of x, x^2, p, and p^2.

d) Find σ_x and σ_p. Is their product consistent with the uncertainty principle?

2. **Quantum Harmonic Oscillator** $[2 * 6 = 12 \text{ Points}]$
Consider the one dimensional quantum harmonic oscillator hamiltonian given by:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2, \tag{2}$$

with $\hat{p}$ and $\hat{x}$ are the momentum and position operators. The *ladder* operators a and $a^\dagger$ are defined as:

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} + \frac{i}{m\omega}\hat{p}), \tag{3}$$

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} - \frac{i}{m\omega}\hat{p}). \tag{4}$$

In the subsequent we drop the hat $(\hat{})$ notation to denote operators.

a) Find the commutator $[a, a^\dagger]$ and write the Hamiltonian in term of the number operator $N = a^\dagger a$.

The number operator is hermitian (convince yourself). This means it can be diagonalized and has real eigenvalues. Let us denote the eigenvectors of N as $|n\rangle$, with the corresponding eigenvalue n, i.e. $N|n\rangle = n|n\rangle$.

b) Find how the hamiltonian H acts on the states $a^\dagger|n\rangle$ and $a|n\rangle$. From this result, deduce to which eigenstates $|n'\rangle$ the previous states are proportional to:

$$a^\dagger|n\rangle = c_+|n'_1\rangle \tag{5}$$

$$a|n\rangle = c_-|n'_2\rangle \tag{6}$$

c) Find c_+ and c_-.

d) So far we have not made any assumption about the numbers n apart from being real. Show that n must be positive or 0. (Hint: consider the norm of $a\left|n\right\rangle$).

e) The ground state of the harmonic oscillator hamiltonian corresponds to the eigenvector $\left|0\right\rangle$. Previously we (should have) found $a\left|0\right\rangle = 0$. Rewriting this equation in the space representation we can find the form of the ground state wave function:

$$a\psi_0(x) = 0. \tag{7}$$

Find $\psi_0(x)$.

f) Find $\langle x\rangle$, $\langle x^2\rangle$, $\langle p\rangle$, $\langle p^2\rangle$, $\langle V\rangle$ and $\langle T\rangle$ for the $\left|n\right\rangle$ state of the harmonic oscillator. T and V are the kinetic and potential energy respectively. Check that the uncertainty principle is satisfied. (Hint: there is no need to do integrals)