Freie Universität Berlin
Tutorials for Advanced Quantum Mechanics
Wintersemester 2018/19
Sheet 3

Due date: 10:15 09/11/2018 J. Eisert

1. **Non-Interacting Identical Particles**(2×2 points)
Consider a system of N non-interacting identical particles whose Hamiltonian has the form

$$H_0 = \sum_{i=1}^{N} H_0(i). \tag{1}$$

Assume the spectrum of $H_0(i)$ to be discrete, and let $\{\psi_{a_j}(i)\}|_j$ be the normalized single-particle eigenfunctions (i.e. the *orbitals*) corresponding to eigenstates of $H_0(i)$ with eigenvalues $\{E_{a_j}\}|_j$, i.e.

$$H_0(i)\psi_{a_j}(i) = E_{a_j}\psi_{a_j}(i). \tag{2}$$

   (a) Consider a fermionic system with $N = 3$. Write down the wave function $\psi_{a_j a_k a_l}(1,2,3)$ - i.e. the wave function for a fermionic system with occupation numbers $N_j = N_k = N_l = 1$. By expanding this wave function, show explicitly that $\psi_{a_j a_j a_l}(1,2,3) = 0$, i.e. that the wave function vanishes if the occupation number of the orbital a_j is $N_j = 2$.

   (b) For a bosonic system with N particles, the basis wave functions are given by totally symmetric wave functions of the form

$$\psi^S_{a_{j_1} a_{j_2} \dots a_{j_N}}(1,2,\dots,N) = C \sum_{P \in S_N} \psi_{a_{j_1}}(P1)\psi_{a_{j_2}}(P2)\dots\psi_{a_{j_N}}(PN) \tag{3}$$

   prove that the correct normalization factor is

$$C = \left[\frac{1}{N!N_1!N_2!\dots}\right]^{1/2} \tag{4}$$

   where N_j is the occupation number of orbital a_j.

2. **Identical Particles: Perfect Gas**(2 × 2 points)
A perfect gas is system with N identical non-interacting particles. In thermal equilibrium, the properties of these gases can be derived from the thermodynamic potential:

$$\Omega = \sum_i \Omega_i, \tag{5}$$

$$\Omega_i = -k_B T \log \sum_{N_i} \left[e^{(\mu - E_i)/k_B T)}\right]^{N_i}, \tag{6}$$

with k_B the Boltzmann constant, T the absolute temperature, μ the chemical potential per particle, and Ω_i the thermodynamic potential of the orbital a_i with energy E_i. The sum $\sum_{N_i}$ runs over all the possible occupation numbers of the orbital a_i. The average number of particles in the orbital a_i is given by $\bar{N}_i = -\frac{\partial \Omega_i}{\partial \mu}$. Calculate $\bar{N}_i$ for:

(a) An ideal fermionic gas.

(b) An ideal bosonic gas.

3. **Many-particle ground states** (2 × 3 points)
Consider spin-1/2 fermions of mass m subject to the potential

$$V(r) = \frac{1}{2}m\omega^2(x^2 + y^2 + z^2) - \mu. \tag{7}$$

If the particle number $N = 1$, the ground state is twofold degenerate and the ground-state energy is $E_1 = (3/2)\hbar\omega - \mu$. The two ground state wavefunctions are.

$$\psi_\uparrow(\mathbf{r}, \sigma) = \frac{1}{(l\sqrt{\pi})^{3/2}} e^{-(x^2+y^2+z^2)/2l^2} \delta_{\sigma,\uparrow}, \tag{8}$$

$$\psi_\downarrow(\mathbf{r}, \sigma) = \frac{1}{(l\sqrt{\pi})^{3/2}} e^{-(x^2+y^2+z^2)/2l^2} \delta_{\sigma,\downarrow}. \tag{9}$$

with $l = \sqrt{\hbar/m\omega}$.

(a) The ground state for N = 2 is non-degenerate. What is its energy? Give an explicit expression for the ground state wavefunction $\psi(\mathbf{r}_1, \sigma_1; \mathbf{r}_2, \sigma_2)$. If you prefer, you may use the bra/ket notation for the spin degree of freedom instead of the notation used above.

(b) The particle number N = 2 is called "magic", because the ground state is non-degenerate at that particle number. What is the magic particle number that comes next after $N = 2$. Explain your answer.

(c) What is the ground state energy and the ground state degeneracy if $N = 2$ and the particles are spin-0 Bosons instead of spin-1/2 fermions?

4. **Four particles in one dimension** (2 × 3 points)
Four one-dimensional non-interacting particles of mass m are confined to a length L with periodic boundary conditions. The Hamiltonian $\hat{H}$ for one particle reads:

$$\hat{H} = \frac{\hat{p}^2}{2m}. \tag{10}$$

The single particle states have wavefunctions:

$$\psi_n(x) = \frac{1}{\sqrt{L}} e^{2\pi i n x/L}, \quad \text{with } n = 0, \pm 1, \pm 2... \tag{11}$$

The creation operator for a particle in a state with wavefunction $\psi_n(x)$ is denoted $\hat{a}_n^\dagger$ for bosons and $\hat{a}_{n,\sigma}^\dagger$ for spin 1/2 fermions with spin $\sigma =\uparrow, \downarrow$.

(a) What are the ground state energy and the ground state degeneracy if the four particles are distinguishable?

(b) What are the ground state energy and the ground state degeneracy if the four particles are indistinguishable spin-0 bosons? Express the ground state(s) in terms of the vacuum state $|0\rangle$ and the creation operators $\hat{a}_n^\dagger$.

(c) What are the ground state energy and the ground state degeneracy if the four particles are indistinguishable spin-1/2 fermions? Express the ground state(s) in terms of the vacuum state $|0\rangle$ and the creation operators $\hat{a}_{n,\sigma}^\dagger$.