

**Freie Universität Berlin**  
**Tutorials for Advanced Quantum Mechanics**  
**Wintersemester 2018/19**  
**Sheet 4**

Due date: 10:15 16/11/2018

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1. **Bosonic and Fermionic commutation relations**(3×2 points)

- (a) Recalling the quantum harmonic oscillator, it is now apparent that the ladder operators,

$$\hat{a} = \frac{1}{\sqrt{2\hbar}}(\hat{x} + i\hat{p}), \quad \hat{a}^\dagger = \frac{1}{\sqrt{2\hbar}}(\hat{x} - i\hat{p}) \quad (1)$$

Using these relations, derive the original form of Heisenbergs uncertainty principle, which states that the standard deviation product of position and momentum measurements is lower bounded as,

$$\Delta\hat{x}\Delta\hat{p} \geq \frac{\hbar}{2} \quad (2)$$

- (b) Starting from the fermionic anti-commutation relations

$$\{\hat{f}_j, \hat{f}_k^\dagger\} = \delta_{j,k}, \quad \{\hat{f}_j, \hat{f}_k\} = \{\hat{f}_j^\dagger, \hat{f}_k^\dagger\} = 0 \quad (3)$$

derive the action of the fermionic creation and annihilation operators on the occupation number basis states,

$$\hat{f}_j |N_1, \dots, N_j, \dots\rangle = (-1)^{\sum_{k=1}^{j-1} N_k} N_j |N_1, \dots, 1 - N_j, \dots\rangle \quad (4)$$

$$\hat{f}_j^\dagger |N_1, \dots, N_j, \dots\rangle = (-1)^{\sum_{k=1}^{j-1} N_k} (1 - N_j) |N_1, \dots, 1 - N_j, \dots\rangle \quad (5)$$

- (c) Consider the single particle Hamiltonian  $\hat{H}_0$  with eigenstates  $\{|\lambda\rangle\}$  - i.e.  $\hat{H}_0|\lambda\rangle = \lambda|\lambda\rangle$ . Let  $|\lambda_1, \dots, \lambda_N\rangle_{B(F)}$  be the corresponding bosonic (fermionic)  $N$  particle basis state in a first quantization representation. We define the number operator as  $\hat{n}_\lambda = \hat{a}_\lambda^\dagger \hat{a}_\lambda$ . Now, by using the second quantization representation of  $|\lambda_1, \dots, \lambda_N\rangle_{B(F)}$ , and the appropriate commutation relations for  $\hat{a}_\lambda^\dagger, \hat{a}_\lambda$ , prove that the number operator  $\hat{n}_\lambda$  simply counts the number of particles in state  $|\lambda\rangle$  - i.e. show explicitly that for both bosonic and fermionic  $N$  particle states

$$\hat{n}_\lambda |\lambda_1, \dots, \lambda_N\rangle_{B(F)} = \sum_{i=1}^N \delta_{\lambda\lambda_i} |\lambda_1, \dots, \lambda_N\rangle_{B(F)} \quad (6)$$

2. Observables in second quantization (2 + 2 + 2 + 4 + 4 points)

- (a) Consider a system of  $N$  particles, and a one-body operator  $\hat{O}_1 = \sum_{j=1}^N \hat{o}_j$ , where  $\hat{o}_j$  is an ordinary single particle operator acting on the  $j$ 'th particle. Furthermore, using the same notation as (1c), assume that  $\hat{O}_1$  is diagonal in the  $\{|\lambda\rangle\}$  basis, i.e.  $\hat{o} = \sum_{\lambda} o_{\lambda} |\lambda\rangle\langle\lambda|$ . Show that a second quantization representation of  $\hat{O}_1$ , with respect to the  $\{|\lambda\rangle\}$  basis, is given by

$$\hat{O}_1 = \sum_{\lambda=0}^{\infty} o_{\lambda} \hat{n}_{\lambda} = \sum_{\lambda=0}^{\infty} \langle\lambda|\hat{o}|\lambda\rangle \hat{a}_{\lambda}^{\dagger} \hat{a}_{\lambda} \quad (7)$$

- (b) What is the second quantized representation of  $\hat{O}_1$  in a different basis  $\{|\mu\rangle\}$ , in which  $\hat{O}_1$  is not diagonal?
- (c) Consider a single particle in a one-dimensional infinite square well of length  $L$ . Write down the basis transformations between  $\hat{a}_p$  and  $\hat{a}(x)$  - i.e. the operators which annihilate a particle at a fixed momentum or position.
- (d) Now consider a many-particle finite one-dimensional system of length  $L$  - i.e a system with infinite square well potential. The single particle kinetic energy operator is given by  $\hat{T} = \sum_j \hat{p}_j^2 / 2m$ . Show that the second quantized representation of this operator is given by

$$\hat{T} = \int_0^L dx \hat{a}^{\dagger}(x) \frac{p^2}{2m} \hat{a}(x) \quad (8)$$

[Hint: Use the strategy developed in (a) and (b), with the tools from (c) - i.e. first express the kinetic energy operator in the basis in which it is diagonal, obtain the second quantized representation in this basis, and then transform into the co-ordinate basis carefully.]

- (e) Consider a bosonic Hamiltonian  $H = \sum_{i,j} h_{i,j} \hat{b}_i^{\dagger} \hat{b}_j$ , with  $\hat{b}_i^{\dagger}, \hat{b}_j$  the usual bosonic creation and annihilation operators. Prove that the Heisenberg picture evolved creation and annihilation operators are given by:

$$\hat{b}_i(t) = \sum_j (e^{-ith})_{i,j} \hat{b}_j \quad (9)$$

$$\hat{b}_i^{\dagger}(t) = \sum_j (e^{ith})_{i,j} \hat{b}_j^{\dagger} \quad (10)$$

[Hint: Again it helps to consider a basis in which the Hamiltonian is diagonal.]