Freie Universität Berlin
Tutorials for Advanced Quantum Mechanics
Wintersemester 2018/19
Sheet 4

Due date: 10:15 16/11/2018 J. Eisert

1. **Bosonic and Fermionic commutation relations**(3×2 points)

   (a) Recalling the quantum harmonic oscillator, it is now apparent that the ladder operators,

   $$\hat{a} = \frac{1}{\sqrt{2\hbar}}(\hat{x} + i\hat{p}), \quad \hat{a}^\dagger = \frac{1}{\sqrt{2\hbar}}(\hat{x} - i\hat{p}) \tag{1}$$

   Using these relations, derive the original form of Heisenbergs uncertainty principle, which states that the standard deviation product of position and momentum measurements is lower bounded as,

   $$\Delta\hat{x}\Delta\hat{p} \geq \frac{\hbar}{2} \tag{2}$$

   (b) Starting from the fermionic anti-commutation relations

   $$\{\hat{f}_j, \hat{f}_k^\dagger\} = \delta_{j,k}, \quad \{\hat{f}_j, \hat{f}_k\} = \{\hat{f}_j^\dagger, \hat{f}_k^\dagger\} = 0 \tag{3}$$

   derive the action of the fermionic creation and annihilation operators on the occupation number basis states,

   $$\hat{f}_j|N_1, \ldots, N_j, \ldots\rangle = (-1)^{\sum_{k=1}^{j-1} N_k} N_j |N_1, \ldots, 1 - N_j, \ldots\rangle \tag{4}$$

   $$\hat{f}_j^\dagger|N_1, \ldots, N_j, \ldots\rangle = (-1)^{\sum_{k=1}^{j-1} N_k} (1 - N_j)\, |N_1, \ldots, 1 - N_j \ldots\rangle \tag{5}$$

   (c) Consider the single particle Hamiltonian $\hat{H}_0$ with eigenstates $\{|\lambda\rangle\}$ - i.e. $\hat{H}_0|\lambda\rangle = \lambda|\lambda\rangle$. Let $|\lambda_1, \ldots \lambda_N\rangle_{B(F)}$ be the corresponding bosonic (fermionic) N particle basis state in a first quantization representation. We define the number operator as $\hat{n}_\lambda = \hat{a}_\lambda^\dagger \hat{a}_\lambda$. Now, by using the second quantization representation of $|\lambda_1, \ldots \lambda_N\rangle_{B(F)}$, and the appropriate commutation relations for $\hat{a}_\lambda^\dagger, \hat{a}_\lambda$, prove that the number operator $\hat{n}_\lambda$ simply counts the number of particles in state $|\lambda\rangle$ - i.e. show explicitly that for both bosonic and fermionic N particle states

   $$\hat{n}_\lambda|\lambda_1, \ldots \lambda_N\rangle_{B(F)} = \sum_{i=1}^{N} \delta_{\lambda\lambda_i} |\lambda_1, \ldots \lambda_N\rangle_{B(F)} \tag{6}$$

2. **Observables in second quantization** (2 + 2 + 2 + 4 + 4 points)

(a) Consider a system of N particles, and a one-body operator $\hat{O}_1 = \sum_{j=1}^{N} \hat{o}_j$, where $\hat{o}_j$ is an ordinary single particle operator acting on the j'th particle. Furthermore, using the same notation as (1c), assume that $\hat{O}_1$ is diagonal in the $\{|\lambda\rangle\}$ basis, i.e. $\hat{o} = \sum_\lambda o_\lambda |\lambda\rangle\langle\lambda|$. Show that a second quantization representation of $\hat{O}_1$, with respect to the $\{|\lambda\rangle\}$ basis, is given by

$$\hat{O}_1 = \sum_{\lambda=0}^{\infty} o_\lambda \hat{n}_\lambda = \sum_{\lambda=0}^{\infty} \langle\lambda|\hat{o}|\lambda\rangle \hat{a}_\lambda^\dagger \hat{a}_\lambda \tag{7}$$

(b) What is the second quantized representation of $\hat{O}_1$ in a different basis $\{|\mu\rangle\}$, in which $\hat{O}_1$ is not diagonal?

(c) Consider a single particle in one-dimensional system of length L with periodic boundary conditions. Write down the basis transformations between $\hat{a}_p$ and $\hat{a}(x)$ - i.e. the operators which annihilate a particle at a fixed momentum or position.

(d) Now consider a many-particle finite one-dimensional system of length L with periodic boundary conditions. The single particle kinetic energy operator is given by $\hat{T} = \sum_j \hat{p_j}^2/2m$. Show that the second quantized representation of this operator is given by

$$\hat{T} = \int_0^L dx \hat{a}^\dagger(x) \frac{\hat{p}^2}{2m} \hat{a}(x) \tag{8}$$

[Hint: Use the strategy developed in (a) and (b), with the tools from (c) - ie. first express the kinetic energy operator in the basis in which it is diagonal, obtain the second quantized representation in this basis, and then transform into the co-ordinate basis carefully.]

(e) Consider a bosonic Hamiltonian $H = \sum_{i,j} h_{i,j} \hat{b}_i^\dagger \hat{b}_j$, with $\hat{b}_i^\dagger, \hat{b}_j$ the usual bosonic creation and annihilation operators. Prove that the Heisenberg picture evolved creation and annihilation operators are given by:

$$\hat{b}_i(t) = \sum_j (e^{-ith})_{i,j} \hat{b}_j \tag{9}$$

$$\hat{b}_i^\dagger(t) = \sum_j (e^{ith})_{i,j} \hat{b}_j^\dagger \tag{10}$$

[Hint: Again it helps to consider a basis in which the Hamiltonian is diagonal.]