**Freie Universität Berlin**
**Tutorials for Advanced Quantum Mechanics**
**Wintersemester 2018/19**
**Sheet 7**

**Due date:** 10:15 7/12/2018 J. Eisert

1. **Unitarity Bogoliubov transformation**(2+4+2 points)
In the lectures you used the following Bogoliubov transformations to diagonalise the Hamiltonian of a weakly interacting Bose gas.

$$\begin{aligned} b_k &= u_k a_k + v_k a^\dagger_{-k} \\ b^\dagger_k &= u_k a^\dagger_k + v_k a_{-k} \end{aligned} \tag{1}$$

where

$$u_k^2 - v_k^2 = 1 \tag{2}$$

In general, Bogoliubov transformations are not necessarily unitary, however many of the most useful ones are.

(a) Consider the operator $U = \exp(\lambda_k(a_k a_{-k} - a^\dagger_{-k} a^\dagger_k))$ with $\lambda \in \mathbb{R}$. Check that setting $u_k = \cosh \lambda_k$ and $v_k = \sinh \lambda_k$ automatically satisfies (2) and that U is unitary ($UU^\dagger = \mathbb{I}$).

(b) Show that

$$U a_k U^\dagger = b_k, \quad U a^\dagger_k U^\dagger = b^\dagger_k \tag{3}$$

i.e. that the unitary $U$ implements the desired Bogoliubov transformation. [Hint: note that the second relation in (3) follows easily given the first and the useful identity $e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + ...$]

(c) Show that $|\varnothing\rangle_b$, the vacuum state for the new operators $b_k$, is related to the vacuum state for $a_k$ via the $|\varnothing\rangle_b = U|\varnothing\rangle_a$

2. **Photon correlations**(2+2+4+2 points)
You already saw in lectures that bosons can exhibit a "clustering" effect where the probability of detecting two bosons at shorter distances was larger than finding them far apart. This effect can be investigated further in the simplest case of 2 spinless bosons. The general 2 boson state can be written as,

$$|2\rangle = \int dx_1 dx_2 \ \ \phi(x_1, x_2) \hat{a}^\dagger(x_1) \hat{a}^\dagger(x_2) |0\rangle \tag{4}$$

(a) Show that the normalisation condition for this state is

$$1 = \langle 2|2\rangle = \int dx_1 dx_2 \phi^* (x_1, x_2) (\phi (x_1, x_2) + \phi (x_2, x_1)) \tag{5}$$

(b) Assume that the wave function is factorisable, $\phi(x_1, x_2) = \sqrt{N}\phi_1(x_1)\phi_2(x_2)$, with the individual $\phi$ normalised so that $\int dx \phi_i(x)^* \phi_i(x) = 1$. Note that it could still be the case that the functions $\phi_1$ and $\phi_2$ overlap (this is the key point). Show that the normalisation condition (5) now implies,

$$\phi(x_1, x_2) = \frac{\phi_1(x_1)\phi_2(x_2)}{\sqrt{1 + |(\phi_1, \phi_2)|^2}} \tag{6}$$

where $(\phi_1, \phi_2) = \int dx \phi_1^*(x)\phi_2(x)$ is the inner product of the two wave functions.

(c) Show that the particle density is $\langle 2|\hat{n}(x)|2\rangle = \langle 2|\hat{a}^\dagger(x)\hat{a}(x)|2\rangle$ is given by,

$$\langle 2|\hat{n}(x)|2\rangle = |\phi_1(x)|^2 + |\phi_2(x)|^2 + (\phi_1, \phi_2)\, \phi_2^*(x)\phi_1(x) + (\phi_2, \phi_1)\, \phi_1^*(x)\phi_2(x) \tag{7}$$

where for orthogonal wave packets, $(\phi_1, \phi_2) = 0$ one recovers the particle density for independent particles, $\langle 2|\hat{n}(x)|2\rangle = |\phi_1(x)|^2 + |\phi_2(x)|^2$ but in general one also has an interference term due to overlapping wave packets.

(d) Now consider overlapping Gaussian wave packets with a separation $a$ described by wave-functions,

$$\phi_1(x) = \frac{1}{\pi^{1/4}} \mathrm{e}^{-\frac{1}{2}(x-a)^2}, \quad \phi_2(x) = \frac{1}{\pi^{1/4}} \mathrm{e}^{-\frac{1}{2}(x+a)^2} \tag{8}$$

Calculate $\langle 2|\hat{n}(x)|2\rangle$ and show that the integrated density gives the value it should, i.e. that $\int dx \langle 2|\hat{n}(x)|2\rangle = 2$. Using some mathematical software, plot $\langle 2|\hat{n}(x)|2\rangle$ along with the density for independent particles $|\phi_1(x)|^2 + |\phi_2(x)|^2$ for large separation (a=4) and small separation (a=1). Explain your results in terms of overlapping bosonic wave functions.