Freie Universität Berlin
Tutorials for Advanced Quantum Mechanics
Wintersemester 2018/19
Sheet 11

Due date: 10:15 18/01/2019 J. Eisert

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1. **Effective Models**(2 + 4 + 4 + 4 points)

   In the theory of superconductivity we encountered a model of interacting fermions, with the interaction mediated by bosonic degrees of freedom. The effective model, however, no longer contained bosonic degrees of freedom. In this exercise we want to get a sense of how such effective models can be derived. To this end, consider a system with two fermionic modes and a single bosonic mode, and consider the Hamiltonian contributions

$$H_1 = f_1 \otimes b^\dagger + f_1^\dagger \otimes b, \tag{1}$$
$$H_2 = f_2 \otimes b^\dagger + f_2^\dagger \otimes b, \tag{2}$$
$$H_3 = 2b^\dagger b - 3f_1^\dagger f_1 - 7f_2^\dagger f_2 \tag{3}$$

   where f_1 and f_2 are the annihilation operators of the fermionic modes and b is the annihilation operator of the bosonic mode. Note that the two fermionic modes are directly coupled to the bosonic mode, but not to each other. We want to prove the existence of an effective interaction between the fermionic modes, mediated via the bosonic mode.

   (a) Write down the ground state of H_3. (2 points)
   (b) Consider the Hamiltonian $H = H_3 + \lambda(H_1 + H_2)$. Using perturbation theory calculate the ground state $|G\rangle$ of H up to first order in λ. (4 points)
   (c) Find the reduced quantum state ρ_{f_1,f_2} (density matrix) for the fermionic degrees of freedom by tracing out the bosonic degree of freedom from $|G\rangle$. (4 points)
   (d) Show that ρ_{f_1,f_2} is not a product state - i.e. that the two fermionic modes are correlated. (4 points)

   The correlations present between the two fermionic modes, despite no direct interaction, is evidence of an effective interaction mediated via the bosonic mode. On a higher level, a similar mechanism is at work in the theory of superconductivity.

2. **Jordan-Wigner Transformation**(4 + 4 points)

   In lectures we saw that systems with n fermionic modes are ismorphic to systems with n spins via the Jordan wigner transformation, which maps fermionic operators to non-local spin operators via

$$f_j^\dagger = \Big(\prod_{k=1}^{j-1} Z_k\Big)\sigma_j^-, \tag{4}$$

$$f_j = \Big(\prod_{k=1}^{j-1} Z_k\Big)\sigma_j^+, \tag{5}$$

where $\sigma_j^- = (X_j - iY_j)/2$ and $\sigma_j^+ = (X_j + iY_j)/2$.

(a) Prove that the non-local spin operators isomorphic to f_j and $f_j^\dagger$ satisfy the fermionic anti-commutation relations - i.e. prove that for any $n \geq 2$

$$\{f_j, f_k\} = \{f_j^\dagger, f_k^\dagger\} = 0 \tag{6}$$

$$\{f_j, f_k^\dagger\} = \delta_{jk} \tag{7}$$

for all $j, k \leq n$. (4 points)

(b) The Jordan-Wigner transformation is non-local in the sense that local fermionic operators are mapped onto non-local spin operators containing strings of Pauli Z operators. We want to prove that this is necessarily the case - i.e. that no Jordan-Wigner type transformation respecting locality can exist. To be more concrete, consider a system consisting of two spins, each with Hilbert space $\mathcal{H}_j = \mathbb{C}^2$, and show that no spin operators a_1 and a_2, with support only on site 1 and 2 respectively, can be constructed which satisfy fermionic anti-commutation relations, i.e. which satisfy

$$\{a_j, a_k^\dagger\} = \delta_{jk}, \tag{8}$$

for all $j, k \leq 2$. (4 points)