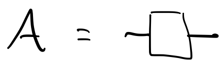
Freie Universität Berlin
Tutorials for Advanced Quantum Mechanics
Wintersemester 2018/19
Sheet 13

Due date: 10:15 01/02/2019 J. Eisert

1. **Tensor Network Notation Warm Up** (7 * 2 = 14 points)

   In general, an N-tensor can be represented by a box with N legs, with each leg representing an index of the tensor. For example, a matrix A can be represented as a box with two legs as follows:

   Contraction over the indices of a tensor is then represented graphically by joining the legs of the corresponding indices. Many linear algebraic operations are expressed very easily in this notation. To see this, draw the tensor network diagram for:

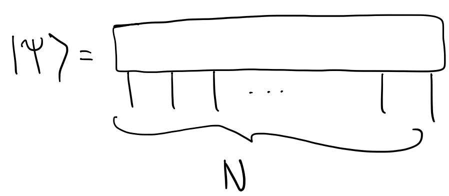
   (a) AB, given matrices A and B.
   (b) $A \otimes B$.
   (c) $A|v\rangle$, where $|v\rangle$ is a vector.
   (d) Tr(AB).
   (e) Tr(A $\otimes$ B).
   (f) $\langle v|A|v\rangle$.
   (g) The singular value decomposition of A.

2. **Matrix Product States** (18 points + 10 bonus points)

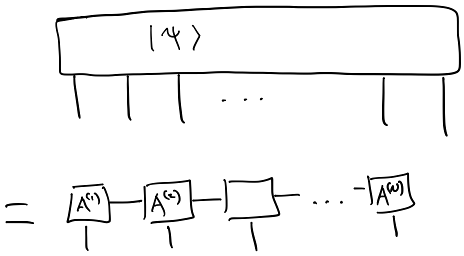
   Consider a multipartite Hilbert space $\mathcal{H} = \bigotimes_{j=1}^{N} \mathcal{H}_j$ with $\mathcal{H}_j \simeq \mathbb{C}^d$. An arbitrary state $|\psi\rangle \in \mathcal{H}$ can be written as

$$|\psi\rangle = \sum_{i_1,\ldots,i_N=1}^{d} \psi_{i_1,\ldots,i_N} |i_1, \ldots, i_N\rangle \quad (1)$$

   where ψ is an N tensor containing all the amplitudes defining the state, represented graphically as follows:

You were told in lectures that this N-tensor can always be decomposed into the following form:



This decomposition is known as a matrix product state (MPS) decomposition of $|\psi\rangle$.

(a) Write down the algebraic expression corresponding to the matrix product state diagram given above - i.e. write out an explicit expression for matrix product state $|\psi\rangle$, in terms of contractions of the 3-tensors $\{A^{(j)}\}$. (2 points)

(b) Motivate the name "matrix product state". (2 points)

(c) !!BONUS!! There exists a canonical procedure for decomposing any 1D state, with open boundary conditions, into a matrix product state – i.e. an explicit procedure for deriving the 3-tensors $\{A^{(j)}\}$ from the N-tensor ψ. This procedure works via sequential reshaping and then singular value decomposing of the original N-tensor ψ . Write down this procedure explicitly - either using tensor network diagrams, or algebraic notation - showing explicitly how to obtain the 3-tensors $\{A^{(j)}\}$. (10 points).

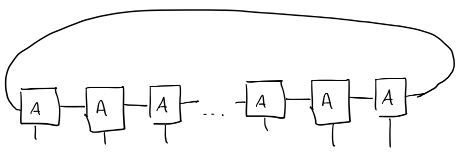
Let's now consider the case of $\mathcal{H}_j = \mathbb{C}^2$, and 1D lattices with periodic boundary conditions. Furthermore, consider the translationally invariant states

$$|0_N\rangle = |000\ldots000\rangle \tag{2}$$

$$|W\rangle = \sum_j |000\ldots1_j\ldots000\rangle \tag{3}$$

$$|GHZ\rangle = |0_N\rangle + |1_N\rangle \tag{4}$$

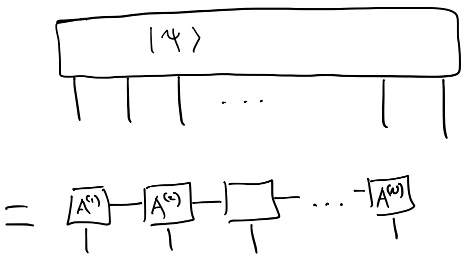
These states can all be represented by tensor networks of the following form.



In particular, as the states are translationally invariant, all the 3-tensors are identical, and each state is fully specified by a single 3-tensor A.

(d) Write down the 3-tensor A explicitly for $|0_N\rangle$, $|W\rangle$ and $|GHZ\rangle$ (2*3=6 points)

Lets fix $N = 6$ and go back to an MPS $|\psi\rangle$ with open boundary conditions, i.e assume $|\psi\rangle$ has been decomposed into the MPS form that we saw before:



Draw the tensor network diagram corresponding to:

(e) $\langle\psi|\psi\rangle$. (2 points)

(f) $\rho = |\psi\rangle\langle\psi|$. (2 points)

(g) $\mathrm{Tr}_A(\rho)$ where $A = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$. (2 points)

(h) $\langle\hat{O}_3\rangle$ where $\hat{O}_3$ is an observable supported only on $\mathcal{H}_3$. (2 points)