1. Representing spins with fermions (4×2 points)
Spin-$\frac{1}{2}$ lattice models are ubiquitous in the study of condensed matter physics and quantum statistical mechanics. In the lectures you learned that a single spin-$\frac{1}{2}$ system can be represented in $\mathbb{C}^2$ by associating the spin-up and spin-down states with vectors $|\uparrow\rangle := \left(\begin{array}{c}1 \\ 0\end{array}\right)$ and $|\downarrow\rangle := \left(\begin{array}{c}0 \\ 1\end{array}\right)$ respectively which form a basis and that measuring in the $\{|\uparrow\rangle, |\downarrow\rangle\}$ basis is described by the Pauli $z$ matrix $\sigma^z$. Measuring whether a particle is spin-up or spin-down is thus described by an operator $S^z = \frac{1}{2}\sigma^z$, i.e. an operator with eigenvalues $\pm\frac{1}{2}$ for the spin-up and spin down states respectively. One can also measure in the real and imaginary spin superposition bases $|\pm\rangle := (|\uparrow\rangle \pm |\downarrow\rangle)/\sqrt{2}$ and $|i\pm\rangle := (|\uparrow\rangle \pm i|\downarrow\rangle)/\sqrt{2}$ with corresponding operators $S^x = \frac{1}{2}\sigma^x$ and $S^y = \frac{1}{2}\sigma^y$. These operators satisfy the canonical commutation relations

$$[S^i, S^j] = i\epsilon_{ijk}S^k$$

(1)

where $\epsilon_{ijk}$ is the Levi-Civita symbol which is 0 if any two indices are the same and $(-1)^{\pi(P)}$ where $\pi(P)$ is the parity of any permutation away from the order $i, j, k = x, y, z$. These commutation relations define the algebra of spin-$\frac{1}{2}$ observables in a similar manner to the bosonic and fermionic case.

For spin ladder operators defined as

$$S^\pm = S^x \pm iS^y$$

(2)

(a) Justify the term spin ladder operators by finding the action of $S^\pm$ on the states $|\uparrow\rangle$ and $|\downarrow\rangle$

(b) Show that

$$\{S^+, S^-\} = 1$$

(3)

and

$$[S^+, S^-] = 2S^z$$

(4)

which is another canonical way of defining the spin algebra.

(c) The anti-commutation relations in (3) and the suggestive names might prompt us to propose a representation of the spin system in terms of fermions by associating the state $|\uparrow\rangle$ with an occupied fermionic particle state $f^\dagger|0\rangle := |1\rangle$ and the state $|\downarrow\rangle$ with the vacuum $f|1\rangle := |0\rangle$. In this representation the spin raising and lowering operators would identified with fermionic creation and annihilation operators via $S^+ = f^\dagger$, $S^- = f$ and $S^z = f^\dagger f - \frac{1}{2}$. Using the fermionic anti-commutation relations $^2$ show that under this definition the spin operators satisfy (4).

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1. The Pauli matrices are given by

$$\sigma^z = \left(\begin{array}{cc}1 & 0 \\ 0 & -1\end{array}\right), \sigma^x = \left(\begin{array}{cc}0 & 1 \\ 1 & 0\end{array}\right), \sigma^y = \left(\begin{array}{cc}0 & -i \\ i & 0\end{array}\right)$$

2. $\{f_j, f_k^\dagger\} = \delta_{jk}$, $\{f_j, f_k\} = \{f_j^\dagger, f_k^\dagger\} = 0$
Consider a 1-D chain of spins with sites labelled \( j = 1, 2, ..., N \) where the \( N \)-site states live in the Hilbert space \( \mathcal{H} = \bigotimes_{j=1}^{N} \mathbb{C}_j^2 \). A spin ladder operator for just one lattice site, \( j \), is given by the corresponding operator defined in (2) (i.e. the original definition in terms of Pauli matrices) on the Hilbert space, \( \mathbb{C}_j^2 \), tensored with the identity on all the others, e.g. \( S_j^+ = 1 \otimes S_j^+ \otimes 1 \otimes ... \otimes 1 \). Given the above results, we might be tempted to represent the spin raising and lowering operators on a site \( j \) with with fermionic creation and annihilation operators for orbitals \( j = 1, 2, ..., N \) via \( S_j^+ = f_j^\dagger \), \( S_j^- = f_j \) and \( S_j^z = f_j^\dagger f_j - \frac{1}{2} \). Explain why the representation breaks down in this case. (Hint: consider the commutator \( [S_1^+, S_2^+] \)).

2. **Representing spins with bosons** (4 × 2 points)

We can also construct representations of the spin algebra in terms of bosons, not only for spin-\( \frac{1}{2} \) systems but for arbitrary spin-\( S \) systems. The state of a spin-\( S \) system is typically written \( |S, m⟩ \) where \( m \) is the \( S_z \) spin component which can take values \(-S, -(S-1), ... S-1, S \). For spin-\( \frac{1}{2} \) particles the only allowed values are \( \pm \frac{1}{2} \), while in general there are \( 2S + 1 \) possible values for \( m \). This state must be an eigenstate of \( S_z \) and also the total spin operator defined as \( S^2 := (S_x)^2 + (S_y)^2 + (S_z)^2 \) satisfying,

\[
S^z |S, m⟩ = m |S, m⟩, \quad S^2 |S, m⟩ = S(S+1) |S, m⟩
\]  
(5)

(a) Using the bosonic commutation relations\(^3\) show that the representation

\[
\hat{S}^- = a^\dagger (2S - a^\dagger a)^{1/2}, \quad \hat{S}^+ = \left( \hat{S}^- \right)^\dagger, \quad \hat{S}^z = S - a^\dagger a
\]  
(6)

where \( a \) and \( a^\dagger \) are bosonic creation and annihilation operators reproduces (4). (Hint: you can prove this without explicitly expanding the square root!)

(b) We now investigate a representation where one spin system is represented by two bosonic modes (or orbitals) defined by operators \((a, a^\dagger)\) and \((b, b^\dagger)\) via the mapping \( \hat{S}^+ = a^\dagger b, \hat{S}^- = \left( \hat{S}^+ \right)^\dagger, \hat{S}^z = \frac{1}{2} (a^\dagger a - b^\dagger b) \).

Show via the bosonic commutation relations that,

\[
|S, m⟩ = \frac{(a^\dagger)^{S+m}}{(S+m)!} \frac{(b^\dagger)^{S-m}}{(S-m)!} |∅⟩
\]  
(7)

satisfy the definitions in (5), where \( |∅⟩ \) is the bosonic vacuum state.

(c) Consider the operators given by the following linear combination of the bosonic operators from the previous question

\[
a_1 = \sqrt{T} a + \sqrt{R} b, \quad b_1 = \sqrt{T} b - \sqrt{R} a
\]  
(8)

with \( T, R \in \mathbb{R} \). Find the conditions on \( T \) and \( R \) such that \( a_1 \) and \( b_1 \) also satisfy the bosonic commutation relations.

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\(^3\)\([a_j, a_k^\dagger] = \delta_{jk}, [a_j, a_k] = [a_j^\dagger, a_k^\dagger] = 0\)
(d) The above transformations describe many physical processes, for example the combination of two spatially distinct beams of light (represented by the modes $a$ and $b$) being combined (or interfered) on a particular kind of glass cube that transmits or reflects each mode in a manner described by the coefficients $T$ and $R$. If we apply this transformation and then measure the operator $a_1^\dagger a_1 - b_1^\dagger b_1$, what does this correspond to in terms of a spin measurement? Based upon your answer, how do we interpret the interference transformation in the spin picture?

3. **Time evolution of the field operators** (4 points)

In lectures you saw that the Hamiltonian for interacting particles in a potential in terms of the field operators was given by,

$$
H = \int d\xi \Psi^\dagger(\xi) \left( -\frac{\hbar^2 \Delta}{2M} + V_1(\xi) \right) \Psi(\xi) + \frac{1}{2} \int d\xi d\xi' \Psi^\dagger(\xi') \Psi^\dagger(\xi) V_2(\xi, \xi') \Psi(\xi) \Psi(\xi')
$$

(9)

Using the Heisenberg equations of motion which simplify in this case to

$$ih \frac{\partial}{\partial t} \Psi(\xi, t) = [H, \Psi(\xi, t)]
$$

(10)

and field operator commutation relations,

$$
[\Psi(\xi), \Psi(\xi')] = [\Psi^\dagger(\xi), \Psi^\dagger(\xi')] = 0
$$

(11)

$$
[\Psi(\xi), \Psi^\dagger(\xi')] = \delta(\xi - \xi')
$$

(12)

and the identity

$$

(13)

show that (10) has the structure of a nonlinear Schrödinger equation, namely

$$
ih \frac{\partial}{\partial t} \Psi(\xi, t) = \left( -\frac{\hbar^2 \Delta}{2M} + V_1(\xi) \right) \Psi(\xi, t) + \int d\xi' \Psi^\dagger(\xi', t) V_2(\xi, \xi') \Psi(\xi', t) \Psi(\xi, t)
$$

(14)