Freie Universität Berlin
Tutorials for Advanced Quantum Mechanics
Wintersemester 2018/19
Sheet 5

Due date: 10:15 23/11/2018 J. Eisert

1. **Representing spins with fermions**(4×2 points)
Spin-$\frac{1}{2}$ lattice models are ubiquitous in the study of condensed matter physics and quantum statistical mechanics. In the lectures you learned that a single spin-$\frac{1}{2}$ system can be represented in $\mathbb{C}^2$ by associating the spin-up and spin-down states with vectors $|\uparrow\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|\downarrow\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ respectively which form a basis and that measuring in the $\{|\uparrow\rangle, |\downarrow\rangle\}$ basis is described by the Pauli z matrix[1] σ^z. Measuring whether a particle is spin-up or spin-down is thus described by an operator $S^z = \frac{1}{2}\sigma^z$, i.e. an operator with eigenvalues $\pm\frac{1}{2}$ for the spin-up and spin down states respectively. One can also measure in the real and imaginary spin superposition bases $|\pm\rangle := (|\uparrow\rangle \pm |\downarrow\rangle)/\sqrt{2}$ and $|i\pm\rangle := (|\uparrow\rangle \pm i|\downarrow\rangle)/\sqrt{2}$ with corresponding operators $S^x = \frac{1}{2}\sigma^x$ and $S^y = \frac{1}{2}\sigma^y$. These operators satisfy the canonical commutation relations

$$[S^i, S^j] = i\epsilon_{ijk}S^k \tag{1}$$

where ϵ_{ijk} is the Levi-Civita symbol which is 0 if any two indices are the same and $(-1)^{\pi(P)}$ where $\pi(P)$ is the parity of any permutation away from the order $i, j, k = x, y, z$. These commutation relations define the algebra of spin-$\frac{1}{2}$ observables in a similar manner to the bosonic and fermionic case.

For spin ladder operators defined as

$$S^{\pm} = S^x \pm iS^y \tag{2}$$

(a) Justify the term *spin ladder operators* by finding the action of $S^{\pm}$ on the states $|\uparrow\rangle$ and $|\downarrow\rangle$

(b) Show that

$$\{S^+, S^-\} = 1 \tag{3}$$

and

$$[S^+, S^-] = 2S^z \tag{4}$$

which is another canonical way of defining the spin algebra.

(c) The anti-commutation relations in (3) and the suggestive names might prompt us to propose a representation of the spin system in terms of fermions by associating the state $|\uparrow\rangle$ with an occupied fermionic particle state $f^\dagger|0\rangle := |1\rangle$ and the state $|\downarrow\rangle$ with the vacuum $f|1\rangle := |0\rangle$. In this representation the spin raising and lowering operators would identified with fermionic creation and annihilation operators via $S^+ = f^\dagger$, $S^- = f$ and $S^z = f^\dagger f - \frac{1}{2}$. Using the fermionic anti-commutation relations[2] show that under this definition the spin operators satisfy (4).

[1] The Pauli matrices are given by $\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

[2] $\{f_j, f_k^\dagger\} = \delta_{jk}$, $\{f_j, f_k\} = \{f_j^\dagger, f_k^\dagger\} = 0$

(d) Consider a 1-D chain of spins with sites labelled $j = 1, 2, ..., N$ where the N-site states live in the Hilbert space $\mathcal{H} = \bigotimes_{j=1}^{N} \mathbb{C}_j^2$. A spin ladder operator for just one lattice site, j, is given by the corresponding operator defined in (2) (i.e. the original definition in terms of Pauli matrices) on the Hilbert space, $\mathbb{C}_j^2$, tensored with the identity on all the others, e.g. $S_2^+ = \mathbb{1} \otimes S^+ \otimes \mathbb{1} \otimes ... \otimes$. Given the above results, we might be tempted to represent the spin raising and lowering operators on a site j with with fermionic creation and annihilation operators for orbitals $j = 1, 2, ..., N$ via $S_j^+ = f_j^\dagger$, $S_j^- = f_j$ and $S_j^z = f_j^\dagger f_j - \frac{1}{2}$. Explain why the representation breaks down in this case. (Hint: consider the commutator $[S_1^+, S_2^+]$)

2. **Representing spins with bosons**(4 × 2 points)
We can also construct representations of the spin algebra in terms of bosons, not only for spin-$\frac{1}{2}$ systems but for arbitrary spin-S systems. The state of a spin-S system is typically written $|S, m\rangle$ where m is the S^z spin component which can take values $-S, -(S-1), ...S-1, S$. For spin-$\frac{1}{2}$ particles the only allowed values are $\pm\frac{1}{2}$, while in general there are $2S + 1$ possible values for m. This state must be an eigenstate of S^z and also the total spin operator defined as $\boldsymbol{S}^2 := (S^x)^2 + (S^y)^2 + (S^z)^2$ satisfying,

$$S^z|S, m\rangle = m|S, m\rangle, \qquad \boldsymbol{S}^2|S, m\rangle = S(S+1)|S, m\rangle \tag{5}$$

(a) Using the bosonic commutation relations[3] show that the representation

$$\hat{S}^- = a^\dagger \left(2S - a^\dagger a\right)^{1/2}, \quad \hat{S}^+ = \left(\hat{S}^-\right)^\dagger, \quad \hat{S}^z = S - a^\dagger a \tag{6}$$

where a and $a^\dagger$ are bosonic creation and annihilation operators reproduces (4). (Hint: you can prove this without explicitly expanding the square root!)

(b) We now investigate a representation where one spin system is represented by two bosonic modes (or orbitals) defined by operators $(a, a^\dagger)$ and $(b, b^\dagger)$ via the mapping $\hat{S}^+ = a^\dagger b, \hat{S}^- = \left(\hat{S}^+\right)^\dagger, \hat{S}^z = \frac{1}{2}\left(a^\dagger a - b^\dagger b\right)$.
Show via the bosonic commutation relations that,

$$|S, m\rangle = \frac{\left(a^\dagger\right)^{S+m}}{\sqrt{(S+m)!}} \frac{\left(b^\dagger\right)^{S-m}}{\sqrt{(S-m)!}} |\varnothing\rangle \tag{7}$$

satisfy the definitions in (5), where $|\varnothing\rangle$ is the bosonic vacuum state.

(c) Consider the operators given by the following linear combination of the bosonic operators from the previous question

$$a_1 = \sqrt{T}a + \sqrt{R}b, \quad b_1 = \sqrt{T}b - \sqrt{R}a \tag{8}$$

with $T, R \in \mathbb{R}$. Find the conditions on T and R such that a_1 and b_1 also satisfy the bosonic commutation relations.

[3] $[a_j, a_k^\dagger] = \delta_{jk}$, $[a_j, a_k] = [a_j^\dagger, a_k^\dagger] = 0$

(d) The above transformations describe many physical processes, for example the combination of two spatially distinct beams of light (represented by the modes a and b) being combined (or interfered) on a particular kind of glass cube that transmits or reflects each mode in a manner described by the coefficients T and R. If we apply this transformation and then measure the operator $a_1^\dagger a_1 - b_1^\dagger b_1$, what does this correspond to in terms of a spin measurement? Based upon your answer, how do we interpret the interference transformation in the spin picture?

3. **Time evolution of the field operators**(4 points)

In lectures you saw that the Hamiltonian for interacting particles in a potential in terms of the field operators was given by,

$$\begin{aligned} H = & \int d\xi \Psi^\dagger(\xi) \left(-\frac{\hbar^2 \Delta}{2M} + V_1(\xi) \right) \Psi(\xi) \\ & + \frac{1}{2} \int d\xi d\xi' \Psi^\dagger(\xi') \Psi^\dagger(\xi) V_2(\xi, \xi') \Psi(\xi) \Psi(\xi') \end{aligned} \tag{9}$$

Using the Heisenberg equations of motion which simplify in this case to

$$i\hbar \frac{\partial}{\partial t} \Psi(\xi, t) = [H, \Psi(\xi, t)] \tag{10}$$

and field operator commutation relations,

$$[\Psi(\xi), \Psi(\xi')] = [\Psi^\dagger(\xi), \Psi^\dagger(\xi')] = 0 \tag{11}$$

$$[\Psi(\xi), \Psi^\dagger(\xi')] = \delta(\xi - \xi') \tag{12}$$

and the identity

$$[AB, C] = A[B, C] + [A, C]B \tag{13}$$

show that (10) has the structure of a nonlinear Schrödinger equation, namely

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \Psi(\xi, t) = & \left(-\frac{\hbar^2 \Delta}{2M} + V_1(\xi) \right) \Psi(\xi, t) \\ & + \int d\xi' \Psi^\dagger(\xi', t) V_2(\xi, \xi') \Psi(\xi', t) \Psi(\xi, t) \end{aligned} \tag{14}$$