1. **Bogoliubov Theory of the Weakly Interacting Bose Gas** (2 + 4 points)

In lectures you utilized the following Bogoliubov transformation as a tool for studying the weakly interacting Bose gas:

\[
\begin{align*}
    b_k &= u_k a_k + v_k a_{-k}^\dagger, \\
    b_k^\dagger &= u_k a_k^\dagger + v_k a_{-k}.
\end{align*}
\]  

(1)

(2)

In order to ensure the operators \( b \) satisfy the Bose commutation relations, it is necessary to enforce

\[ u_k^2 - v_k^2 = 1. \]  

(3)

Additionally, we saw that in order to ensure that non-diagonal terms of the transformed Hamiltonian vanish, it is necessary to enforce

\[
\left( \frac{k^2}{2m} + nV_k \right) u_k v_k + \frac{n}{2} V_k (u_k^2 + v_k^2) = 0.
\]  

(4)

(a) Derive explicitly the inverse of the Bogoliubov transformation given in eqns. (1) and (2).

(b) Equations (3) and (4) specify a system of equations which can be used to solve for \( u_k \) and \( v_k \). Verify explicitly that

\[
\begin{align*}
    u_k^2 &= \frac{w_k + \left( \frac{k^2}{2m} + nV_k \right)}{2w_k}, \\
    v_k^2 &= -w_k + \left( \frac{k^2}{2m} + nV_k \right) = \frac{(nV_k)^2}{2\omega_k (\omega_k + \frac{k^2}{2m} + nV_k)}, \\
    u_kv_k &= \frac{nV_k}{2\omega_k}
\end{align*}
\]

where \( w_k \) is defined as

\[ w_k = \left( \frac{k^2}{2m} + nV_k \right)^2 - \left( nV_k \right)^2 \]  

\[ \frac{1}{2}. \]
2. **Coherent States** (20 points)

Coherent states are a convenient basis when working with bosonic harmonic-oscillator-like Hamiltonians. The dynamics of these coherent states resemble the oscillatory behaviour of a classical harmonic oscillator. They are widely used in quantum optics and laser physics. In this exercise you will derive their properties. Consider the harmonic oscillator Hamiltonian:

\[
\hat{H} = \hbar \omega (\hat{n} + \frac{1}{2}), \quad \text{with} \quad \hat{n} = a^\dagger a.
\]  

(5)

Coherent states are defined as the eigenstates of the annihilation operator \(a\):

\[
a|\alpha\rangle = \alpha|\alpha\rangle,
\]

(6)

where, since \(a\) is not Hermitian, \(\alpha = |\alpha|e^{i\phi}\) is complex.

(a) Find the expectation value \(\bar{n}\) and \(\bar{H}\) of \(\hat{n}\) and \(\hat{H}\) for the coherent state \(|\alpha\rangle\). (2 Points)

(b) Show that the phase shifting operator \(U(\theta) = e^{-i\theta \hat{n}}\) adds a phase to the annihilation operator \(a\): (2 Points)

\[
U(\theta)^\dagger a U(\theta) = ae^{-i\theta}
\]

(7)

and that it shifts the phase of a coherent state:

\[
U(\theta)|\alpha\rangle = |\alpha e^{-i\theta}\rangle.
\]

(8)

(c) Consider now the so called displacement operator: (4 Points)

\[
D(\alpha) = e^{\alpha a^\dagger - \alpha^* a}.
\]

(9)

This operator is the 'creation' operator for the coherent states. Before proving this claim, show that this operator satisfies the following properties:

i. Show that it can also be written as:

\[
D(\alpha) = e^{-\frac{1}{2}|\alpha|^2} e^{a^\dagger \alpha} e^{-\alpha^* a}.
\]

(10)

[Hint: use the property \(e^{A+B} = e^A e^B e^{-[A,B]/2}\) valid when \([A, B], B = [A, B], B = 0\) (check that this is satisfied in our case)].

ii. Show that is a unitary operator:

\[
D(\alpha)D(\alpha)^\dagger = 1,
\]

(11)

Also check that \(D(\alpha)^\dagger = D(-\alpha)\).

iii. Show that:

\[
D(\alpha)^\dagger a D(\alpha) = a + \alpha.
\]

(12)

iv. Finally show that:

\[
D(\alpha)|0\rangle = |\alpha\rangle,
\]

(13)

where \(|0\rangle\) is the vacuum (\(a|0\rangle = 0\)).
(d) The coherent states can be expressed in terms of the eigenstates $|n\rangle$ of the number operator $\hat{n}$: (6 Points)

$$|\alpha\rangle = \sum |n\rangle \langle n|\alpha\rangle.$$  \hspace{1cm} (14)

i. Find an expression for $\langle n|\alpha\rangle$ in terms of $\langle 0|\alpha\rangle$. [Hint: use the defining equation of the coherent states (6)].

ii. Find $\langle 0|\alpha\rangle$.

iii. Show that:

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{(\alpha a^\dagger)^n}{n!} |0\rangle.$$  \hspace{1cm} (15)

(e) Calculate the probability of finding $n$ bosons in the state $|\alpha\rangle$ and express that in terms of the expectation value $\bar{n}$. You should find a Poisson distribution. (2 Points)

(f) To finish show that the basis of coherent states is complete: (4 Points)

$$\frac{1}{\pi} \int d^2\alpha |\alpha\rangle \langle \alpha| = 1,$$  \hspace{1cm} (16)

where the integration is done over the complex plane. [Hint: use the completeness of the number basis $\sum |n\rangle \langle n| = 1$]

**Final remark:** coherent states are not orthogonal $\langle \alpha|\beta\rangle \neq 0$. You can check this as an optional exercise.