1. Details of BCS Theory (4 + 4 + 4 points)

In lectures you saw the following Hamiltonian as a starting point for developing the BCS theory of super-conductivity:

\[ H = H_0 + H_1 \]  
\[ H_0 = \sum_{k,\sigma} \epsilon_k f_{k,\sigma}^{\dagger} f_{k,\sigma} \]  
\[ H_1 = -\frac{1}{2V} \sum_{k,k'} V_{k,k'} f_{k,\sigma}^{\dagger} f_{-k',-\sigma} + f_{-k,-\sigma} f_{k',\sigma} \]

with fermionic operator \( f_{k,\sigma}^{\dagger} \) creating an electron with wave number \( k \) and spin-component \( \sigma \).

As in previous settings, and according to a general theme, in order to diagonalize this Hamiltonian it is convenient to introduce new operators \( A_k \) and \( B_k \) via

\[ f_{k,1/2} = u_k A_k + v_k B_k^{\dagger} \]  
\[ f_{-k,-1/2} = u_k B_k - v_k A_k^{\dagger} \]

where \( u_k \) and \( v_k \) are real functions satisfying \( u_k = u_{-k} \), \( v_k = v_{-k} \) and \( u_k^2 + v_k^2 = 1 \).

In lectures it was claimed that the following Hamiltonian could then be obtained via the above transformation:

\[ H = E_0 + H_0' + H_1' + H_2' \]  
\[ E_0 = 2 \sum_k \epsilon_k v_k^2 - \frac{1}{V} \sum_{k,k'} V_{k,k'} u_k v_k u_{k'} v_{k'} \]  
\[ H_0' = \sum_k \left( \epsilon_k (u_k^2 - v_k^2) + \frac{2u_k v_k}{V} \sum_{k'} V_{k,k'} u_{k'} v_{k'} \right) \times (A_k^{\dagger} A_k + B_k^{\dagger} B_k) \]  
\[ H_1' = \sum_k \left( 2\epsilon_k u_k v_k - \frac{(u_k^2 - v_k^2)}{V} \sum_{k'} V_{k,k'} u_{k'} v_{k'} \right) \times (A_k^{\dagger} B_k + A_k B_k) \]

where \( H_2' \) contains higher order terms whose contribution to computation of the lowest energies is negligible. Again, and in accordance with a general strategy, in order to diagonalise the transformed Hamiltonian (6) we use the degrees of freedom we have introduced in eqs. (4) and (5) in order to set \( H_1' = 0 \). If we take
\[ u_k = \frac{1}{\sqrt{2}} \left( 1 + \frac{\epsilon_k}{\sqrt{\Delta_k^2 + \epsilon_k^2}} \right)^{1/2} \]  

(10)

\[ v_k = \frac{1}{\sqrt{2}} \left( 1 - \frac{\epsilon_k}{\sqrt{\Delta_k^2 + \epsilon_k^2}} \right)^{1/2} \]  

(11)

then it was claimed in lectures that \( H_1' = 0 \) as long as \( \Delta_k \) is the solution to the equation

\[ \Delta_k = \frac{1}{2V} \sum_{k'} \frac{V_{k,k'} \Delta_{k'}}{\sqrt{\Delta_{k'}^2 + \epsilon_{k'}^2}} \]  

(12)

(a) Prove that the operators \( A_k \) and \( B_k \) satisfy fermionic commutation relations, given the constraints on \( u_k \) and \( v_k \). (4 Points)

(b) Use these commutation relations to derive explicitly the Hamiltonian (6), by substituting (4) and (5) into the original Hamiltonian (1). (4 points)

(c) Given eqs. (10) and (11), prove explicitly that eq. (12) is the equation that \( \Delta_k \) should satisfy in order to set \( H_1' = 0 \). (4 points)