Freie Universität Berlin
Advanced Quantum Mechanics
Wintersemester 2018/19

Exercise sheet 1

Due: 25.10.2019 10:15 J. Eisert

1. **Wave function basics** $[2+2+2+2=8 \text{ Points}]$
A particle of mass m has as a wave function given by:

$$\Psi(x,t) = Ae^{-a(mx^2/\hbar + it)}, \tag{1}$$

where the constants A and a are positive and real.

(a) Find A such that $\Psi(x,t)$ is normalized.

(b) Consider a Hamiltonian of the form $\hat{H} = \frac{\hat{p}^2}{2m} + V(x)$. Find the potential energy function $V(x)$ for which $\Psi(x,t)$ satisfies the Schrödinger equation, i. e. is an eigenstate of $\hat{H}$.

(c) Calculate the expectation values $\langle x \rangle_\Psi$, $\langle x^2 \rangle_\Psi$, $\langle p \rangle_\Psi$, and $\langle p^2 \rangle_\Psi$.

(d) Find σ_x and σ_p. Is their product consistent with the uncertainty principle?

2. **Quantum Harmonic Oscillator** $[\sum_a^f 2 = 12 \text{ Points}]$
Consider the one dimensional quantum harmonic oscillator Hamiltonian given by:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2, \tag{2}$$

with $\hat{p}$ and $\hat{x}$ are the momentum and position operators. The *ladder* operators a and $a^\dagger$ are defined as:

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} + \frac{i}{m\omega}\hat{p}), \tag{3}$$

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} - \frac{i}{m\omega}\hat{p}). \tag{4}$$

In the subsequent we drop the hat (ˆ) notation to denote operators.

(a) Find the commutator $[a, a^\dagger]$ and write the Hamiltonian in terms of the number operator $N = a^\dagger a$.

The number operator is hermitian (convince yourself). This means it can be diagonalized and has real eigenvalues. Let us denote the eigenvectors of N as $|n\rangle$, with the corresponding eigenvalue n, i.e. $N|n\rangle = n|n\rangle$.

(b) Find how the Hamiltonian H acts on the states $a^\dagger |n\rangle$ and $a|n\rangle$. From this result, deduce to which eigenstates $|n'\rangle$ the previous states are proportional to:

$$a^\dagger |n\rangle = c_+ |n'_1\rangle \tag{5}$$

$$a|n\rangle = c_- |n'_2\rangle \,, \tag{6}$$

where c_+ and c_- are constants.

(c) Find c_+ and c_-.

(d) So far we have not made any assumption about the numbers n apart from being real. Show that n must be positive or 0. (Hint: consider the norm of $a\,|n\rangle$).

(e) The ground state of the harmonic oscillator Hamiltonian corresponds to the eigenvector $|0\rangle$. Previously we (should have) found $a\,|0\rangle = 0$. Rewriting this equation in the space representation we can find the form of the ground state wave function:

$$a\psi_0(x) = 0. \tag{7}$$

Find $\psi_0(x)$.

(f) Find $\langle x\rangle$, $\langle x^2\rangle$, $\langle p\rangle$, $\langle p^2\rangle$, $\langle V\rangle$ and $\langle T\rangle$ for the $|n\rangle$ state of the harmonic oscillator. T and V are the kinetic and potential energy respectively. Check that the uncertainty principle is satisfied. (Hint: there is no need to do integrals)

3. **Entropy and entanglement** $[2+3+2+3 = 10$ Points$]$
The entropy of a quantum state is an important quantity, often used to study entanglement in many-body quantum systems. Given a density matrix ρ, it is defined as

$$S(\rho) := -\mathrm{tr}(\rho \ln \rho). \tag{8}$$

A useful fact to note is that $0 \leq S(\rho) \leq \ln(d)$, where d is the dimension of the system.

(a) When are the above inequalities saturated?

(b) Let $|\Phi\rangle_{\mathrm{AB}} = \frac{1}{\sqrt{2}}[|01\rangle_{\mathrm{AB}} - |10\rangle_{\mathrm{AB}}]$. This is known as the singlet state. What is $S(|\Phi\rangle\langle\Phi|_{\mathrm{AB}})$? What is $S(\rho_A)$ and $S(\rho_B)$, where ρ_A and ρ_B are the reduced density matrices of $|\Phi\rangle\langle\Phi|_{\mathrm{AB}}$ for A, B respectively?

(c) Given a density matrix ρ, show that its transpose ρ^T is also a valid density matrix. This means that the transpose operation, namely $T(M) = M^T$ is a *positive* operation, i.e. it maps density matrices to density matrices. You may use the fact that $\det(A) = \det(A^T)$ for all matrices A.

(d) The *partial transpose* is the operation that performs the transpose operation T, but only on part of the density matrix. More concretely, given the expression of ρ_{AB} as $\rho_{AB} = \sum_{ijkl} c_{ijkl}\,|i\rangle\langle j|_A \otimes |k\rangle\langle j|_B$, then

$$T_B(\rho_{AB}) = \sum_{ijkl} c_{ijkl}\,|i\rangle\langle j|_A \otimes T(|k\rangle\langle l|_B) = \sum_{ijkl} c_{ijkl}\,|i\rangle\langle j|_A \otimes |l\rangle\langle k|_B\,. \tag{9}$$

Use $|\Phi\rangle_{AB}$ to show that the partial transpose T_B is not positive, namely it does not map density matrices to density matrices. In fact, this operation is often used to check whether the system is entangled (as is $|\Phi_{\mathrm{AB}}\rangle$)!