Freie Universität Berlin
Tutorials for Advanced Quantum Mechanics
Wintersemester 2018/19
Sheet 12

Due date: 10:15 24/01/2019 J. Eisert

1. **Entanglement in Many-Body Systems** (30 points)

Soon in lectures you will be introduced to tensor network states, a powerful tool for representing and understanding many-body quantum states. The goal of this question is to introduce some of the fundamental tools you need to understand and work with tensor network states.

To start off, consider a bipartite Hilbert space $\mathcal{H} = \mathcal{H}_\mathcal{A} \otimes \mathcal{H}_B$ with $\mathcal{H}_A \simeq \mathbb{C}^d$ and $\mathcal{H}_B \simeq \mathbb{C}^d$. Let $\{|i\rangle\}|_{i=1}^d$ and $\{|j\rangle\}|_{j=1}^d$ be a basis for $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively.

(a) Write down an expression for an arbitrary state $|\psi\rangle \in \mathcal{H}$ in terms of the above given basis vectors for $\mathcal{H}_A$ and $\mathcal{H}_B$. (0 points)

(b) Starting from your answer to (a), prove that there always exists orthonormal bases $\{|\lambda_i^A\rangle\}|_{i=1}^d$ and $\{|\lambda_j^B\rangle\}|_{j=1}^d$ for $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively, and real positive numbers $\{\lambda_k\}_{k=1}^r$, such that any arbitrary state $|\psi\rangle \in \mathcal{H}$ can equivalently be written as

$$|\psi\rangle = \sum_{k=1}^{r} \lambda_k |\lambda_k^A\rangle|\lambda_k^B\rangle. \tag{1}$$

This representation of $|\psi\rangle$ is called the Schmidt decomposition of $|\psi\rangle$, and r is called the Schmidt rank of $|\psi\rangle$. What is an upper bound on r? (6 points)

(c) Starting from the Schmidt decomposition of $|\psi\rangle$ write down the reduced density matrices ρ_A and ρ_B. How can the Schmidt co-efficients $\{\lambda_k\}$ be interpreted? (2 points)

(d) Write down an expression for a maximally entangled state $|\phi\rangle \in \mathcal{H}$. What are the Schmidt coefficients of this state? (2 points)

Given a state $|\psi\rangle \in \mathcal{H}$, where $\mathcal{H} = \mathcal{H}_\mathcal{A} \otimes \mathcal{H}_B$ is a bipartite Hilbert space, the *entanglement entropy* of $|\psi\rangle$, with respect to the bipartition between A and B, is defined as

$$S_{A|B}(|\psi\rangle) = -\mathrm{Tr}(\rho_A \log_2 \rho_A), \tag{2}$$

where $\rho = |\psi\rangle\langle\psi|$.

(e) Using insight from your answer to (c), write down an alternative expression for $S_{A|B}$ in terms of ρ_B. (2 points)

(f) Write down an alternative expression for $S_{A|B}$ in terms of the Schmidt coefficients $\{\lambda_k\}$ of $|\psi\rangle$. (2 points)

(g) What is the entanglement entropy of the maximally entangled state $|\phi\rangle$? What is the entanglement entropy of the product state $|\psi\rangle = |j\rangle \otimes |k\rangle$? (2 points)

Let's now consider a tripartite Hilbert space $\mathcal{H} = \mathcal{H}_\mathcal{A} \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ with $\mathcal{H}_X \simeq \mathbb{C}^2$ for all $X \in \{A, B, C\}$. Additionally, consider the following states

$$|\xi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \otimes |0\rangle \tag{3}$$

$$|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle) \tag{4}$$

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) \tag{5}$$

(h) Write down $S_{A|BC}(|\tau\rangle)$ and $S_{AB|C}(|\tau\rangle)$ for all $|\tau\rangle \in \{|\xi\rangle, |W\rangle, |GHZ\rangle\}$. (4 points)

Finally, let's go back to a bipartite Hilbert space $\mathcal{H} = \mathcal{H}_\mathcal{A} \otimes \mathcal{H}_B$ and consider two states with the same Schmidt coefficients, i.e.

$$|\psi\rangle = \sum_{k=1}^{r} \lambda_k |\lambda_k^A\rangle|\lambda_k^B\rangle \quad \text{and} \quad |\tau\rangle = \sum_{k=1}^{r} \lambda_k |w_k^A\rangle|w_k^B\rangle. \tag{6}$$

(i) Show that $|\psi\rangle$ and $|\tau\rangle$ are related by a local unitary transformation - i.e. by a unitary of the form $U \otimes V$ with U and V unitary. Write down this unitary explicitly. (4 points)

(j) Show explicitly that any local unitary transformation leaves the Schmidt coefficients invariant. Given what you have shown about the connection between entanglement and Schmidt coefficients, provide some intutition about why this has to be true. (6 points)