Freie Universität Berlin
Tutorials for Advanced Quantum Mechanics
Wintersemester 2019/20
Sheet 3

Due date: 10:15 08.11.2019 J. Eisert

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1. **Non-Interacting Identical Particles** (2+2 = 4 points)
Consider a system of N non-interacting identical particles whose Hamiltonian has the form
$$H_0 = \sum_{i=1}^{N} h_i.$$
Assume the spectrum of h_i to be discrete, and let $\{\psi_i(\xi_i)\}$ be the normalized single-particle eigenfunctions (i.e. the *orbitals*) corresponding to eigenstates of h_i with eigenvalues $\{E_i\}$, i.e.
$$h_i\psi_i(\xi_i) = E_i\psi_i(\xi_i).$$
   (a) Consider a fermionic system with $N = 3$. Write down the wave function $\psi_{j,k,l}(\xi_i, \xi_j, \xi_k)$ By expanding this wave function, show explicitly that $\psi_{j,k,l}(\xi_j, \xi_k, \xi_l) = 0$, if $\xi_j = \xi_k$. What is the name of this effect?
   (b) For a bosonic system with N particles, the basis wave functions are given by totally symmetric wave functions of the form
$$\psi_N^S(\xi_1, \xi_2, \dots, \xi_N) = C \sum_{P \in S_N} \psi_1(\xi_{P(1)})\psi_2(\xi_{P(2)}) \dots \psi_N(\xi_{P(N)})$$
prove that the correct normalization factor is
$$C = \left[\frac{1}{N!N_1!N_2!\dots}\right]^{1/2}$$
where N_j is the occupation number of orbital j.

2. **Many-particle ground states** (2+2+2 = 6 points)
Consider spin-1/2 fermions of mass m subject to the potential
$$V(r) = \frac{1}{2}m\omega^2(x^2 + y^2 + z^2) - \mu.$$
If the particle number $N = 1$, the ground state is twofold degenerate and the ground-state energy is $E_1 = (3/2)\hbar\omega - \mu$. The two ground state wavefunctions are.
$$\psi_\uparrow(\mathbf{r}, \sigma) = \frac{1}{(l\sqrt{\pi})^{3/2}} e^{-(x^2+y^2+z^2)/2l^2} \delta_{\sigma,\uparrow},$$
$$\psi_\downarrow(\mathbf{r}, \sigma) = \frac{1}{(l\sqrt{\pi})^{3/2}} e^{-(x^2+y^2+z^2)/2l^2} \delta_{\sigma,\downarrow}.$$
with $l = \sqrt{\hbar/m\omega}$.

(a) The ground state for N = 2 is non-degenerate. What is its energy? Give an explicit expression for the ground state wavefunction $\psi(\mathbf{r}_1, \sigma_1; \mathbf{r}_2, \sigma_2)$. If you prefer, you may use the bra/ket notation for the spin degree of freedom instead of the notation used above.

(b) The particle number N = 2 is called "magic", because the ground state is non-degenerate at that particle number. What is the magic particle number that comes next after $N = 2$. Explain your answer.

(c) What is the ground state energy and the ground state degeneracy if $N = 2$ and the particles are spin-0 Bosons instead of spin-1/2 fermions?

3. **Free fermionic hopping problem** (0+2+6+6+2+2+2 = 20 points)

A standard problem in quantum many-body theory is the hopping problem. Consider a one-dimensional lattice of length N with periodic boundary conditions. This means that the space is discretized into N sites and the $N+1$ and the first site are identified. Hint: For many of the tasks you might find it useful to employ the discrete Fourier transform of the Kronecker delta: $\delta_{kl} = \frac{1}{N}\sum_{j=1}^{N} e^{2\pi \mathrm{i}(k-l)\frac{j}{N}}$.

(a) Make a sketch of the geometry of the problem.

The Hamiltonian we want to consider is given by

$$H = -\sum_{j=1}^{N}(f_j^\dagger f_{j+1} + f_{j+1}^\dagger f_j)$$

where $f_j^\dagger, f_j$ are the creation and annihilation operators acting on site j. Assume the fermions to be spinless. To diagonalize this Hamiltonian, we introduce new operators $c_k = \frac{1}{\sqrt{N}}\sum_{j=1}^{N} e^{2\pi \mathrm{i}k\frac{j}{N}} f_j$ and $c_k^\dagger = \frac{1}{\sqrt{N}}\sum_{j=1}^{N} e^{-2\pi \mathrm{i}k\frac{j}{N}} f_j^\dagger$. This is actually also a discrete Fourier transform.

(b) Show that the new operators $c_k^\dagger, c_k$ are valid fermionic creation and annihilation operators by calculating their anti-commutator.

(c) Invert the relations for $c_k^\dagger, c_k$ and plug the resulting operators into the Hamiltonian to show that it is diagonalized by the new operators and find the energy as a function of momentum k.

(d) What is the energy of the many-body ground state of this system and which states are occupied? (If you did not solve the above question, assume the energy to be $E(k) = -\cos(2\pi k/N)$, where k is the discrete momentum.)

(e) Assume you have the same energy function for spinful fermions and that the spin states are degenerate. What would the many-body ground state of such a system look like and what is its energy?

(f) Assume you have the same energy function for bosons. What would the many-body ground state of such a system look like and what is its energy?