Freie Universität Berlin
Tutorials for Advanced Quantum Mechanics
Wintersemester 2019/20
Sheet 4

Due date: 10:15 15/11/2019 J. Eisert

Disclaimer: due to the late posting of this tutorial, the tutors will accept any submission before 18/11/2019 10.00hrs without invoking any penalties in grading.

1. **Bosonic and Fermionic commutation relations** (4+2+4+4 = 14 points)

   (a) Convince yourself of the fact, that the variance of an operator $\hat{A}$ can be written as
   $$\sigma_A^2 = \langle(\hat{A} - \langle\hat{A}\rangle)\Psi|\,|(\hat{A} - \langle\hat{A}\rangle)\Psi\rangle$$
   Use this identity to prove that
   $$\sigma_A^2\sigma_B^2 \geq \left(\frac{1}{2\mathrm{i}}\left\langle\left[\hat{A},\hat{B}\right]\right\rangle\right)^2$$

   (b) Combining the result from (a) and the ladder operators for the quantum harmonic oscillator given by
   $$\hat{a} = \frac{1}{\sqrt{2\hbar}}(\hat{x} + i\hat{p}), \quad \hat{a}^\dagger = \frac{1}{\sqrt{2\hbar}}(\hat{x} - i\hat{p})\,,$$
   derive the original form of Heisenbergs uncertainty principle, which states that the standard deviation product of position and momentum measurements is lower bounded as
   $$\sigma_x\sigma_p \geq \frac{\hbar}{2}$$

   (c) Starting from the fermionic anti-commutation relations
   $$\{\hat{f}_j, \hat{f}_k^\dagger\} = \delta_{j,k}, \quad \{\hat{f}_j, \hat{f}_k\} = \{\hat{f}_j^\dagger, \hat{f}_k^\dagger\} = 0$$
   derive the action of the fermionic creation and annihilation operators on the occupation number basis states,
   $$\begin{aligned}\hat{f}_j|N_1,\ldots,N_j,\ldots\rangle &= (-1)^{\sum_{k=1}^{j-1}N_k}N_j|N_1,\ldots,1-N_j,\ldots\rangle\\ \hat{f}_j^\dagger|N_1,\ldots,N_j,\ldots\rangle &= (-1)^{\sum_{k=1}^{j-1}N_k}(1-N_j)\,|N_1,\ldots,1-N_j\ldots\rangle\end{aligned}$$

   (d) Consider the single particle Hamiltonian $\hat{H}_0$ with eigenstates $\{|\lambda\rangle\}$ - i.e. $\hat{H}_0|\lambda\rangle = \lambda|\lambda\rangle$. Let $|\lambda_1,\ldots\lambda_N\rangle_{B(F)}$ be the corresponding bosonic (fermionic) N particle basis state in a first quantization representation. We define the number operator as $\hat{n}_\lambda = \hat{a}_\lambda^\dagger\hat{a}_\lambda$. Now, by using the second quantization representation of $|\lambda_1,\ldots\lambda_N\rangle_{B(F)}$, and the appropriate commutation relations for $\hat{a}_\lambda^\dagger, \hat{a}_\lambda$, prove that the number operator $\hat{n}_\lambda$ simply counts the number of particles in state $|\lambda\rangle$ - i.e. show explicitly that for both bosonic and fermionic N particle states
   $$\hat{n}_\lambda|\lambda_1,\ldots\lambda_N\rangle_{B(F)} = \sum_{i=1}^{N}\delta_{\lambda\lambda_i}|\lambda_1,\ldots\lambda_N\rangle_{B(F)}$$

2. **Observables in second quantization** (2+2+2+4+6 = 16 points)

(a) Consider a system of N particles, and a one-body operator $\hat{O}_1 = \sum_{j=1}^{N} \hat{o}_j$, where $\hat{o}_j$ is an ordinary single particle operator acting on the j'th particle. Furthermore, using the same notation as (1c), assume that $\hat{O}_1$ is diagonal in the $\{|\lambda\rangle\}$ basis, i.e. $\hat{o} = \sum_\lambda o_\lambda |\lambda\rangle\langle\lambda|$. Show that a second quantization representation of $\hat{O}_1$, with respect to the $\{|\lambda\rangle\}$ basis, is given by

$$\hat{O}_1 = \sum_{\lambda=0}^{\infty} o_\lambda \hat{n}_\lambda = \sum_{\lambda=0}^{\infty} \langle\lambda|\hat{o}|\lambda\rangle \hat{a}_\lambda^\dagger \hat{a}_\lambda$$

(b) What is the second quantized representation of $\hat{O}_1$ in a different basis $\{|\mu\rangle\}$, in which $\hat{O}_1$ is not diagonal?

(c) Consider a single particle in a one-dimensional infinite square well of length L. Write down the basis transformations between $\hat{a}_p$ and $\hat{a}(x)$ - i.e. the operators which annihilate a particle at a fixed momentum or position.

(d) Now consider a many-particle finite one-dimensional system of length L - i.e a system with infinite square well potential. The single particle kinetic energy operator is given by $\hat{T} = \sum_j \hat{p_j}^2/2m$. Show that the second quantized representation of this operator is given by

$$\hat{T} = \int_0^L dx \hat{a}^\dagger(x) \frac{p^2}{2m} \hat{a}(x)$$

[Hint: Use the strategy developed in (a) and (b), with the tools from (c) - ie. first express the kinetic energy operator in the basis in which it is diagonal, obtain the second quantized representation in this basis, and then transform into the co-ordinate basis carefully.]

(e) Consider a bosonic Hamiltonian $H = \sum_{i,j} h_{i,j} \hat{b}_i^\dagger \hat{b}_j$, with $\hat{b}_i^\dagger, \hat{b}_j$ the usual bosonic creation and annihilation operators. Prove that the Heisenberg picture evolved creation and annihilation operators are given by:

$$\hat{b}_i(t) = \sum_j (e^{-ith})_{i,j} \hat{b}_j$$

$$\hat{b}_i^\dagger(t) = \sum_j (e^{ith})_{i,j} \hat{b}_j^\dagger$$

[Hint: Again it helps to consider a basis in which the Hamiltonian is diagonal.]