Freie Universität Berlin
Tutorials for Advanced Quantum Mechanics
Wintersemester 2019/20
Sheet 7

Due date: 10:15 6/12/2019 J. Eisert

1. **Unitarity Bogoliubov transformation** (3+5+2 = 10 points)
In the lectures you used the following Bogoliubov transformations to diagonalise the Hamiltonian of a weakly interacting Bose gas.

$$\begin{aligned} b_k &= u_k a_k + v_k a^\dagger_{-k} \\ b^\dagger_k &= u_k a^\dagger_k + v_k a_{-k} \end{aligned} \tag{1}$$

where

$$u_k^2 - v_k^2 = 1 \tag{2}$$

In general, Bogoliubov transformations are not necessarily unitary, however many of the most useful ones are.

(a) Consider the operator $U = \exp(\lambda_k(a_k a_{-k} - a^\dagger_{-k} a^\dagger_k))$ with $\lambda \in \mathbb{R}$. Check that setting $u_k = \cosh\lambda_k$ and $v_k = \sinh\lambda_k$ automatically satisfies (2) and that U is unitary ($UU^\dagger = \mathbb{I}$).

(b) Show that

$$U a_k U^\dagger = b_k, \quad U a^\dagger_k U^\dagger = b^\dagger_k \tag{3}$$

i.e. that the unitary U implements the desired Bogoliubov transformation. [Hint: note that the second relation in (3) follows easily given the first and the useful identity $e^A B e^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + ...$]

(c) Show that $|\varnothing\rangle_b$, the vacuum state for the new operators b_k, is related to the vacuum state for a_k via the $|\varnothing\rangle_b = U|\varnothing\rangle_a$

2. **A quantum quench** (2+2+4+5+2+2+3 = 20 points)

Here, we will investigate a quantum quench, which is an abrupt change of the Hamiltonian governing the system creating an out-of-equilibrium situation. We will investigate whether the system afterwards relaxes to equilibrium again, which is an important question for quantum statistical mechanics.

In classical mechanics, the system equilibrates by the mechanism of mixing.

(a) Show that in quantum mechanics, the purity (as a measure of mixedness) is preserved under unitary time evolution.

In the following, we will investigate whether a quantum system can still attain equilibrium. Let us consider a non-interacting bosonic system and calculate its local particle number expectation value over time. Consider an initial state given by $|\Psi\rangle = \prod_{j=1}^{L/2} b_{2j}^\dagger |\varnothing\rangle = |0,1,0,1,0,1,\ldots\rangle$.

(b) Show that the expectation value $\langle b_j^\dagger b_k \rangle_{|\Psi\rangle}$ evaluates to $\delta_{j,k}\delta_{j \bmod 2,0}$, where mod is the modulo division.

We will now introduce the Hamiltonian given by $H = -\sum_j \left(b_j^\dagger b_{j+1} + b_{j+1}^\dagger b_j \right)$.

(c) Using your results from Sheet 3 and 4, show that the time evolution of $b_j^\dagger$ and b_j is given by

$$b_j(t) = \frac{1}{L} \sum_{\substack{p=1,\\ m=1}}^{L} e^{-\mathrm{i}t\lambda_p} e^{2\pi \mathrm{i}p(j-m)/L} b_m, \qquad b_j^\dagger(t) = \frac{1}{L} \sum_{\substack{q=1,\\ n=1}}^{L} e^{\mathrm{i}t\lambda_q} e^{-2\pi \mathrm{i}p(j-n)/L} b_n^\dagger,$$

where $\lambda_p = -2\cos(\frac{2\pi p}{L})$ is the energy of the p-th level.

In what follows, we will calculate the time evolution of the particle number on a site j for the given initial state $|\Psi\rangle$. This is in close analogy to a research paper (arxiv.org/abs/0808.3779, see appendices A and C).

(d) Show that the time evolved expectation value of the particle number operator acting on site j, $N_j(t) = \langle b_j^\dagger(t) b_j(t) \rangle_{|\Psi\rangle}$, can be written as

$$N_j(t) = \frac{1}{L} \sum_{p,q}^{L} e^{\mathrm{i}t(\lambda_q - \lambda_p)} e^{2\pi \mathrm{i}j(q-p)/L} \underbrace{\frac{1}{L} \sum_m e^{2\pi \mathrm{i}(2m-1)(q-p)/L}}_{(*)} .$$

To obtain this, use your result from (b).

(e) Now prove that $(*) = \delta_{q,p} - \delta_{|q-p|,L/2}$.

(f) Now you are in position to show that

$$N_j(t) = \frac{1}{2} - \frac{(-1)^j}{2L} \sum_{p=1}^{L} e^{4\mathrm{i}t\cos(2\pi p/L)}$$

by plugging in the result from (e) into the equation from (d).
(Do that :] You can even do so if you did not solve the tasks above.)

(g) Taking the limit $L \to \infty$, the expression yields

$$N_j(t) = \frac{1}{2} - \frac{(-1)^j}{2} J_0(4t),$$

where J_0 is the Bessel function of first kind. Plot the resulting time resolved particle number for odd and even sites. Admire the relaxation of the particle number to an equilibrium value.