Freie Universität Berlin
Tutorials for Advanced Quantum Mechanics
Wintersemester 2018/19
Sheet 5

Due date: 10:15 22/11/2019 J. Eisert

1. **Holstein-Primakoff transform** (2+2+3+3 = 10 points)
In this exercise we construct representations of the spin-S algebra in terms of bosons. The state of a spin-S system is typically written $|S,m\rangle$ where m is the S^z spin component which can take values $-S, -(S-1), ...S-1, S$. For spin-$\frac{1}{2}$ particles the only allowed values are $\pm\frac{1}{2}$, while in general there are $2S+1$ possible values for m. This state must be an eigenstate of S^z and also the total spin operator defined as $\boldsymbol{S}^2 := (S^x)^2 + (S^y)^2 + (S^z)^2$ satisfying,

$$S^z|S,m\rangle = m|S,m\rangle, \qquad \boldsymbol{S}^2|S,m\rangle = S(S+1)|S,m\rangle \tag{1}$$

(a) Using the bosonic commutation relations[1] show that the representation

$$\hat{S}^- = a^\dagger \left(2S - a^\dagger a\right)^{1/2}, \quad \hat{S}^+ = \left(\hat{S}^-\right)^\dagger, \quad \hat{S}^z = S - a^\dagger a\,,$$

where a and $a^\dagger$ are bosonic creation and annihilation operators fulfills

$$[S^+, S^-] = 2S^z\,.$$

(Hint: you can prove this without explicitly expanding the square root!)

(b) We now investigate another representation where one spin system is represented by two bosonic modes (or orbitals) defined by operators $(a, a^\dagger)$ and $(b, b^\dagger)$ via the mapping

$$\hat{S}^+ = a^\dagger b, \quad \hat{S}^- = \left(\hat{S}^+\right)^\dagger \quad \text{and} \quad \hat{S}^z = \frac{1}{2}\left(a^\dagger a - b^\dagger b\right)\,.$$

Show via the bosonic commutation relations that,

$$|S,m\rangle = \frac{\left(a^\dagger\right)^{S+m}}{\sqrt{(S+m)!}}\frac{\left(b^\dagger\right)^{S-m}}{\sqrt{(S-m)!}}|\varnothing\rangle\,,$$

satisfy the definitions in equation (1), where $|\varnothing\rangle$ is the bosonic vacuum state.

(c) Consider the operators given by the following linear combination of the bosonic operators from the previous question

$$a_1 = \sqrt{T}a + \sqrt{R}b, \quad b_1 = \sqrt{T}b - \sqrt{R}a\,,$$

with $T, R \in \mathbb{R}$. Find the conditions on T and R such that a_1 and b_1 also satisfy the bosonic commutation relations. The above transformations describe many physical processes, for example the combination of two spatially distinct beams of light (represented by the modes a and b) being combined (or interfered) on a particular kind of glass cube that transmits or reflects each mode in a manner described by the coefficients T (transmission) and R (reflection).

[1] $[a_j, a_k^\dagger] = \delta_{jk}$, $[a_j, a_k] = [a_j^\dagger, a_k^\dagger] = 0$

(d) Suppose we did interfere two bosons resulting in the above transformation. Calculate the boson number difference $a_1^\dagger a_1 - b_1^\dagger b_1$ in terms of a spin measurement. Based upon your answer, how do we interpret the interference transformation in the spin picture?

2. **Jordan-Wigner transform** (2+2+2+2+2+4 = 14 points)
In this exercise we will explore how fermionic systems can be mapped onto spin-$\frac{1}{2}$ systems. Let us define the spin operators

$$S^x = \frac{1}{2}\sigma^x \quad S^y = \frac{1}{2}\sigma^y \quad S^z = \frac{1}{2}\sigma^z\,,$$

These operators satisfy the canonical commutation relations

$$[S^i, S^j] = i\epsilon_{ijk}S^k\,,$$

where ϵ_{ijk} is the Levi-Civita symbol which is 0 if any two indices are the same and $(-1)^{\pi(P)}$ where $\pi(P)$ is the parity of any permutation away from the order $i, j, k = x, y, z$. These commutation relations define the algebra of spin-$\frac{1}{2}$ observables in a similar manner to the bosonic and fermionic case.

Furthermore, we define spin ladder operators as

$$S^{\pm} = S^x \pm iS^y\,.$$

(a) Justify the term *spin ladder operators* by finding the action of $S^{\pm}$ on the states $|\uparrow\rangle$ and $|\downarrow\rangle$.

(b) Show that

$$\{S^+, S^-\} = 1\,,$$

and

$$[S^+, S^-] = 2S^z\,,$$

which is another canonical way of defining the spin algebra.

The anti-commutation relations above and the suggestive names might prompt us to propose a representation of the spin system in terms of fermions by associating the state $|\uparrow\rangle$ with an occupied fermionic particle state $f^\dagger|0\rangle := |1\rangle$ and the state $|\downarrow\rangle$ with the vacuum $f|1\rangle := |0\rangle$. In this representation the spin raising and lowering operators would identified with fermionic creation and annihilation operators via

$$S^+ = f^\dagger, \quad S^- = f \quad \text{and} \quad S^z = f^\dagger f - \frac{1}{2}\,.$$

(c) This time explicitly using the fermionic anti-commutation relations[2] show (again) that $[S^+, S^-] = 2S^z$.

Consider a one dimensional chain of spins with sites labelled $j = 1, 2, ..., N$ where the N-site states live in the Hilbert space $\mathcal{H} = \bigotimes_{j=1}^{N} \mathbb{C}_j^2$. A spin ladder operator for just one lattice site, j, is given by the corresponding operator defined above (namely the original definition in terms of Pauli matrices) on the Hilbert space, $\mathbb{C}_j^2$, tensored with the identity on all the others, e.g. $S_2^+ = \mathbb{1} \otimes S^+ \otimes \mathbb{1} \otimes ...\otimes$.

[2] $\{f_j, f_k^\dagger\} = \delta_{jk}$, $\{f_j, f_k\} = \{f_j^\dagger, f_k^\dagger\} = 0$

Given the above results, we might be tempted to represent the spin raising and lowering operators on a site j with with fermionic creation and annihilation operators for orbitals $j = 1, 2, ..., N$ via $S_j^+ = f_j^\dagger$, $S_j^- = f_j$ and $S_j^z = f_j^\dagger f_j - \frac{1}{2}$.

(d) Explain why the representation breaks down in this case. (Hint: consider the commutator $[S_1^+, S_2^+]$)

To obtain a faithful spin representation, it is necessary cancel this unwanted sign. Although a general procedure is hard to formulate, in one dimension, this can be achieved by a non-linear transformation

$$S_m^+ = f_m^\dagger (-1)^{\sum_{l<m} n_m}, \quad S_m^- = (-1)^{\sum_{l<m} n_m} f_m \quad \text{and} \quad S_m^z = f_m^\dagger f_m - \frac{1}{2}.$$

(e) Use the transformation defined above to show that

$$S_m^+ S_{m+1}^- = f_m^\dagger f_{m+1}.$$

(f) Finally, show that the anisotropic Heisenberg spin chain

$$H = -\sum_n \left(J_z S_n^z S_{n+1}^z + \frac{J_\perp}{2} \left(S_n^+ S_{n+1}^- + S_n^- S_{n+1}^+ \right) \right)$$

can be mapped to the fermionic Hubbard model (when setting $J_z = 0$ this model should be familiar to you from previous exercises. You can start from this case and add in the interaction term later).

3. **Time evolution of the field operators** (3+3+0 = 6 points)

In lectures you saw that the Hamiltonian for interacting particles in a potential in terms of the field operators was given by,

$$\begin{aligned} H &= H^{(1)} + H^{(2)}, \\ H^{(1)} &= \int d\xi' \Psi^\dagger(\xi') \left(-\frac{\hbar^2 \Delta}{2M} + V_1(\xi') \right) \Psi(\xi') \\ H^{(2)} &= \frac{1}{2} \int d\xi' d\xi'' \Psi^\dagger(\xi'') \Psi^\dagger(\xi') V_2(\xi', \xi'') \Psi(\xi') \Psi(\xi'') \end{aligned}$$

The field operator commutation relations are

$$\begin{aligned} [\Psi(\xi), \Psi(\xi')] &= \left[\Psi^\dagger(\xi), \Psi^\dagger(\xi') \right] = 0, \\ \left[\Psi(\xi), \Psi^\dagger(\xi') \right] &= \delta(\xi - \xi'), \end{aligned}$$

and a useful identity for commutators is $[AB, C] = A[B, C] + [A, C]B$.

(a) Using the above tools, show that

$$[H^{(1)}, \Psi(\xi)] = \left(-\frac{\hbar^2 \Delta}{2M} + V_1(\xi) \right) \Psi(\xi).$$

(b) Show that

$$[H^{(2)}, \Psi(\xi)] = -\int d\xi' \Psi^\dagger(\xi') V_2(\xi', \xi) \Psi(\xi') \Psi(\xi).$$

c) Using the Heisenberg equation of motion which simplify in this case to

$$i\hbar\frac{\partial}{\partial t}\Psi(\xi,t) = -[H,\Psi(\xi,t)],$$

show that the equation above has the structure of a nonlinear Schrödinger equation, namely

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\Psi(\xi,t) &= \left(-\frac{\hbar^2\Delta}{2M} + V_1(\xi)\right)\Psi(\xi,t) \\ &+ \int d\xi'\Psi^\dagger\left(\xi',t\right)V_2\left(\xi,\xi'\right)\Psi\left(\xi',t\right)\Psi(\xi,t) \end{aligned}$$