Freie Universität Berlin
Tutorials for Advanced Quantum Mechanics
Wintersemester 2019/20
Sheet 8

Due date: 10:15 13/12/2018 J. Eisert

1. **Photon correlations** (2+2+2+2 = 8 points)
You already saw in lectures that bosons can exhibit a "clustering" effect where the probability of detecting two bosons at shorter distances was larger than finding them far apart. This effect can be investigated further in the simplest case of 2 spinless bosons. The general 2 boson state can be written as,

$$|2\rangle = \int dx_1 dx_2 \;\; \phi(x_1, x_2)\hat{a}^\dagger(x_1)\hat{a}^\dagger(x_2)|0\rangle$$

(a) Show that the normalisation condition for this state is

$$1 = \langle 2|2\rangle = \int dx_1 dx_2 \phi^* (x_1, x_2) \left(\phi (x_1, x_2) + \phi (x_2, x_1)\right)$$

(b) Assume that the wave function is factorisable, $\phi(x_1, x_2) = \sqrt{N}\phi_1(x_1)\phi_2(x_2)$, with the individual ϕ normalised so that $\int dx\phi_i(x)^*\phi_i(x) = 1$. Note that it could still be the case that the functions ϕ_1 and ϕ_2 overlap (this is the key point). Show that the normalisation condition now implies,

$$\phi(x_1, x_2) = \frac{\phi_1(x_1)\phi_2(x_2)}{\sqrt{1 + |(\phi_1, \phi_2)|^2}}$$

where $(\phi_1, \phi_2) = \int dx\phi_1^*(x)\phi_2(x)$ is the inner product of the two wave functions.

(c) Show that the particle density is $\langle 2|\hat{n}(x)|2\rangle = \langle 2|\hat{a}^\dagger(x)\hat{a}(x)|2\rangle$ is given by,

$$\langle 2|\hat{n}(x)|2\rangle = |\phi_1(x)|^2 + |\phi_2(x)|^2 + (\phi_1, \phi_2)\, \phi_2^*(x)\phi_1(x) + (\phi_2, \phi_1)\, \phi_1^*(x)\phi_2(x)$$

where for orthogonal wave packets, $(\phi_1, \phi_2) = 0$ one recovers the particle density for independent particles, $\langle 2|\hat{n}(x)|2\rangle = |\phi_1(x)|^2 + |\phi_2(x)|^2$ but in general one also has an interference term due to overlapping wave packets.

(d) Now consider overlapping Gaussian wave packets with a separation a described by wave-functions,

$$\phi_1(x) = \frac{1}{\pi^{1/4}}\mathrm{e}^{-\frac{1}{2}(x-a)^2}, \quad \phi_2(x) = \frac{1}{\pi^{1/4}}\mathrm{e}^{-\frac{1}{2}(x+a)^2}$$

Calculate $\langle 2|\hat{n}(x)|2\rangle$ and show that the integrated density gives the value it should, i.e. that $\int dx\langle 2|\hat{n}(x)|2\rangle = 2$. Using some mathematical software, plot $\langle 2|\hat{n}(x)|2\rangle$ along with the density for independent particles $|\phi_1(x)|^2 + |\phi_2(x)|^2$ for large separation (a=4) and small separation (a=1). Explain your results in terms of overlapping bosonic wave functions.

2. **Coherent States** ($\sum_a^k 2 = 22$ points)

Coherent states are a convenient basis when working with bosonic harmonic-oscillator-like Hamiltonians. The dynamics of these coherent states resemble the oscillatory behaviour of a classical harmonic oscillator. They are widely used in quantum optics and laser physics. In this exercise you will derive their properties. Consider the harmonic oscillator Hamiltonian:

$$\hat{H} = \hbar\omega(\hat{n} + \frac{1}{2}), \quad \text{with} \quad \hat{n} = a^\dagger a.$$

Coherent states are defined as the eigenstates of the annihilation operator a:

$$|\alpha\rangle = \alpha|\alpha\rangle, \qquad \langle\alpha|\alpha\rangle = 1$$

where, since a is not Hermitian, $\alpha = |\alpha|e^{i\phi}$ is complex.

(a) Find the expectation value of $\hat{n}$ and $\hat{H}$ for the coherent state $|\alpha\rangle$.

(b) Show that the phase shifting operator $U(\theta) = e^{-i\theta\hat{n}}$ adds a phase to the annihilation operator a:

$$U(\theta)^\dagger a U(\theta) = ae^{-i\theta}$$

and that it shifts the phase of a coherent state:

$$U(\theta)|\alpha\rangle = |\alpha e^{-i\theta}\rangle.$$

Consider now the so called displacement operator

$$D(\alpha) = e^{\alpha a^\dagger - \alpha^* a}.$$

This operator is the 'creation' operator for the coherent states. Before proving this claim, show that this operator satisfies the following properties:

(c) Show that it can also be written as:

$$D(\alpha) = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha a^\dagger} e^{-\alpha^* a}.$$

[Hint: use the property $e^{A+B} = e^A e^B e^{-[A,B]/2}$ valid when $[[A,B],B] = [[A,B],B] = 0$ (check that this is satisfied in our case)].

(d) Show that it is a unitary operator:

$$D(\alpha)D(\alpha)^\dagger = \mathbb{1},$$

Also check that $D(\alpha)^\dagger = D(-\alpha)$.

(e) Show that furthermore

$$D(\alpha)^\dagger a D(\alpha) = a + \alpha.$$

(f) And finally show that:

$$D(\alpha)|0\rangle = |\alpha\rangle,$$

where $|0\rangle$ is the vacuum ($a|0\rangle = 0$).

The coherent states can be expressed in terms of the eigenstates $|n\rangle$ of the number operator $\hat{n}$:

$$|\alpha\rangle = \sum |n\rangle\langle n|\alpha\rangle.$$

(g) Find an expression for $\langle n|\alpha\rangle$ in terms of $\langle 0|\alpha\rangle$. [Hint: use the defining equation of the coherent states].

(h) Find $\langle 0|\alpha\rangle$.

(i) Show that:

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{(\alpha a^\dagger)^n}{n!} |0\rangle.$$

(j) Calculate the probability of finding n bosons in the state $|\alpha\rangle$ and express that in terms of the particle number expectation value. You should find a Poisson distribution.

(k) To finish show that the basis of coherent states is complete:

$$\frac{1}{\pi} \int d^2\alpha |\alpha\rangle\langle\alpha| = \mathbb{1},$$

where the integration is done over the complex plane. [Hint: use the completeness of the number basis $\sum |n\rangle\langle n| = \mathbb{1}$]

Final remark: coherent states are not orthogonal $\langle\alpha|\beta\rangle \neq 0$. You can check this as an optional exercise.