Freie Universität Berlin
Tutorials for Advanced Quantum Mechanics
Wintersemester 2018/19
Sheet 9

Due date: 10:15 20/12/2019 J. Eisert

1. **Bogoliubov Theory of weakly interacting Bose gas** ($5 + 5 = 10$ points)
In lectures you utilized the following Bogoliubov transformation as a tool for studying the weakly interacting Bose gas:

$$b_k = u_k a_k + v_k a^\dagger_{-k}, \tag{1}$$
$$b^\dagger_k = u_k a^\dagger_k + v_k a_{-k}. \tag{2}$$

In order to ensure that $b_k, b^\dagger_k$ satisfy the Bose commutation relations, it is necessary that

$$u_k^2 - v_k^2 = 1. \tag{3}$$

Additionally, we saw that in order to ensure that non-diagonal terms of the transformed Hamiltonian vanish, it is necessary to enforce

$$\left(\frac{k^2}{2m} + nV_k\right) u_k v_k + \frac{n}{2} V_k (u_k^2 + v_k^2) = 0. \tag{4}$$

(a) Derive explicitly the inverse of the Bogoliubov transformation in eqns. (1) and (2).

(b) Equations (3) and (4) specify a system of equations which can be used to solve for u_k and v_k. Verify explicitly that

$$u_k^2 = \frac{w_k + \left(\frac{k^2}{2m} + nV_k\right)}{2w_k},$$
$$v_k^2 = \frac{-w_k + \left(\frac{k^2}{2m} + nV_k\right)}{2w_k} = \frac{(nV_k)^2}{2\omega_k(\omega_k + \frac{k^2}{2m} + nV_k)},$$
$$u_k v_k = -\frac{nV_k}{2\omega_k},$$

where $w_k = \sqrt{\left(\frac{k^2}{2m} + nV_k\right)^2 - \left(nV_k\right)^2}$.

2. **Details of BCS Theory**($4 + 4 + 4 + 4 + 4 = 20$ points)
In lectures you saw the following Hamiltonian as a starting point for developing the BCS theory of super-conductivity: $H = H_0 + H_1$, where

$$H_0 = \sum_{k,\sigma} \epsilon_k f^\dagger_{k,\sigma} f_{k,\sigma} \tag{5}$$
$$H_1 = -\frac{1}{2V} \sum_{k,k'} V_{k,k'} f^\dagger_{k,\sigma} f^\dagger_{-k,-\sigma} f_{-k',-\sigma} f_{k',\sigma} \tag{6}$$

with fermionic operator $f^\dagger_{k,\sigma}$ creating an electron with wave number k and spin σ.

As in previous settings, and according to a general theme, in order to diagonalize this Hamiltonian it is convenient to introduce new operators A_k and B_k via

$$f_{k,1/2} = u_k A_k + v_k B_k^\dagger, \qquad f_{-k,-1/2} = u_k B_k - v_k A_k^\dagger \tag{7}$$

where u_k and v_k are real functions satisfying $u_k = u_{-k}$, $v_k = v_{-k}$ and $u_k^2 + v_k^2 = 1$. In lectures it was claimed that the following Hamiltonian could then be obtained via the above transformation:

$$H = E_0 + H_0' + H_1' + H_2' \tag{8}$$

$$E_0 = 2\sum_k \epsilon_k v_k^2 - \frac{1}{V}\sum_{k,k'} V_{k,k'} u_k v_k u_{k'} v_{k'} \tag{9}$$

$$H_0' = \sum_k \Big(\epsilon_k (u_k^2 - v_k^2) + \frac{2u_k v_k}{V}\sum_{k'} V_{k,k'} u_{k'} v_{k'}\Big) \times \big(A_k^\dagger A_k + B_k^\dagger B_k\big) \tag{10}$$

$$H_1' = \sum_k \Big(2\epsilon_k u_k v_k - \frac{(u_k^2 - v_k^2)}{V}\sum_{k'} V_{k,k'} u_{k'} v_{k'}\Big) \times \big(A_k^\dagger B_k^\dagger + A_k B_k\big) \tag{11}$$

where H_2' contains higher order terms whose contribution to computation of the lowest energies is negligible. Again, and in accordance with a general strategy, in order to diagonalise the transformed Hamiltonian (8) we use the degrees of freedom we have introduced in eqs. (7) in order to set $H_1' = 0$. If we take

$$u_k = \frac{1}{\sqrt{2}}\Big(1 + \frac{\epsilon_k}{\sqrt{\Delta_k^2 + \epsilon_k^2}}\Big)^{1/2} \tag{12}$$

$$v_k = \frac{1}{\sqrt{2}}\Big(1 - \frac{\epsilon_k}{\sqrt{\Delta_k^2 + \epsilon_k^2}}\Big)^{1/2} \tag{13}$$

then it was claimed in lectures that $H_1' = 0$ as long as Δ_k is the solution to the equation

$$\Delta_k = \frac{1}{2V}\sum_{k'} \frac{V_{k,k'}\Delta_{k'}}{\sqrt{\Delta_{k'}^2 + \epsilon_{k'}^2}}. \tag{14}$$

(a) Prove that the operators A_k and B_k satisfy fermionic commutation relations, given the constraints on u_k and v_k.

(b) Use these commutation relations to derive explicitly the Hamiltonian (8), by substituting (7) into the original Hamiltonian (5).

(c) Given eqs. (12) and (13), prove explicitly that eq. (14) is the equation that Δ_k should satisfy in order to set $H_1' = 0$.

(d) The BCS ground state vector, as encountered during your lectures, is given by

$$|\psi_{\text{BCS}}\rangle = \prod_k \Big(u_k + v_k P_k^\dagger\Big)\,|\varnothing\rangle, \tag{15}$$

where $P_k^\dagger = f_{k,1/2}^\dagger f_{-k,-1/2}^\dagger$ is known as a Cooper pair. Show that the amplitude

$$\langle\psi_{\text{BCS}}|P_k^\dagger|\psi_{\text{BCS}}\rangle \cdot \langle\psi_{\text{BCS}}|P_k|\psi_{\text{BCS}}\rangle \tag{16}$$

is non-zero.

(e) Verify the commutator

$$\Big[P_k, P_\ell^\dagger\Big] = \delta_{k,\ell}[1 - N_{p,1/2} - N_{-\ell,-1/2}]. \tag{17}$$

In this sense, Cooper pairs are not entirely equivalent to bosons, since they do not satisfy the usual bosonic CR.