Advanced quantum mechanics (20104301)

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Chapter 7: Superconductors

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Chapter 6

Bosonic systems and superfluidity

6.1 Densities and correlations

6.1.1 Density distribution for free bosons

Before coming to these exciting insights, we will, however, start from pretty dry basics. We consider a system of N non-interacting bosons in the state vector

$$|\psi_N\rangle = |N_{p_0}, N_{p_1}, \ldots\rangle. \tag{6.1}$$

The particle number density as a function of position x is

$$\begin{aligned} \langle\psi_N|\Psi^\dagger(x)\Psi(x)|\psi_N\rangle &= \frac{1}{V}\sum_{k,k'} e^{-i(kx+k'x)}\langle\psi_N|b_k^\dagger b_k|\psi_N\rangle \\ &= \frac{N}{V}. \end{aligned} \tag{6.2}$$

This expression is manifestly independent of position, but this is no surprise in a system that is translationally invariant.

6.1.2 Pair distribution function for non-interacting bosons

We can, in analogy to the fermionic case, again define a *pair distribution function*, now in the bosonic reading, without a spin. It is defined as follows.

Bosonic pair distribution function: For spinless bosons, it is defined as

$$\frac{N^2}{V^2} g(x-x') = \langle\psi_N|\Psi^\dagger(x)\Psi^\dagger(x)\Psi(x')\Psi(x')|\psi_N\rangle. \tag{6.3}$$

We also have

$$\begin{aligned}\frac{N^2}{V^2} g(x-x') &= \frac{1}{V^2}\sum_{k,k',q,q'} e^{-ikx-iqx'+iq'x'+ik'x} \\ &\times \langle\psi_N|b_k^\dagger b_q^\dagger b_{q'} b_{k'}|\psi_N\rangle.\end{aligned} \tag{6.4}$$

The expectation value gives a value different from zero only if $k=k'$ and $q=q'$, or if $k=q'$ and $q=k'$. The case of $k=q$ that has been impossible for fermions but that is now possible, has to be considered separately. Hence, we have

$$\begin{aligned}\langle\psi_N|b_k^\dagger b_q^\dagger b_{q'} b_{k'}|\psi_N\rangle &= (1-\delta_{k,q}) \\ &\times \Big(\delta_{k,k'}\delta_{q,q'}\langle\psi_N|b_k^\dagger b_q^\dagger b_q b_k|\psi_N\rangle + \delta_{k,q}\delta_{q,k'}\langle\psi_N|b_k^\dagger b_q^\dagger b_k b_q|\psi_N\rangle\Big) \\ &+ \delta_{k,q}\delta_{k,k'}\delta_{q,q'}\langle\psi_N|b_k^\dagger b_k^\dagger b_k b_k|\psi_N\rangle \\ &= (1-\delta_{k,q}) \\ &\times (\delta_{k,k'}\delta_{q,q'} + \delta_{k,q'}\delta_{q,k'})\, n_k n_q + \delta_{k,q}\delta_{k,k'}\delta_{q,q'} n_k(n_k-1).\end{aligned} \tag{6.5}$$

We therefore arrive at the somewhat complicated looking expression

$$\begin{aligned}&\langle\psi_N|\Psi^\dagger(x)\Psi^\dagger(x')\Psi(x')\Psi(x)|\psi_N\rangle \\ =\;& \frac{1}{V^2}\left(\sum_{k,q}(1-\delta_{k,q})(1+e^{-i(k-q)(x-x')})n_k n_q + \sum_k n_k(n_k-1)\right) \\ =\;& \frac{1}{V^2}\left(\sum_{k,q} n_k n_q - \sum_k n_k^2 + \left|\sum_k e^{-ik(x-x')} n_k\right|^2 - \sum_k n_k^2\right. \\ +\;& \left.\sum_k n_k^2 - \sum_k n_k\right) \\ =\;& \frac{N^2}{V^2} + \frac{1}{V}\left|\sum_k e^{-ik(x-x')} n_k\right|^2 - \frac{1}{V^2}\sum_k n_k(n_k+1).\end{aligned} \tag{6.6}$$

There is a positive second term, reminiscent of the fermionic situation. For the last term, however, there is no fermionic equivalent, simply because one cannot have a double occupation for fermions.

Let us now look at a couple of examples. If all bosons are taking the same momentum state p_0, then (6.6) becomes

$$\frac{N^2}{V^2} g(x-x') = \frac{N(N-1)}{V^2}. \tag{6.7}$$

Now the pair distribution function is also constant. There are simply no correlations The right hand side can be interpreted in a way that the probability to find the first particle N/V ist, that to find the second $(N-1)/V$ and so on.

The situation changes significantly if particles are distributed over momentum values, say, following a Gaussian distribution

$$n_k = \frac{(2\pi)^3 N}{V(\sqrt{\pi}\Delta)^3)} e^{-(k-k_0)^2/\Delta^2} \tag{6.8}$$

with normalization

$$\int \frac{dp}{(2\pi)^3} n_k = \frac{N}{V}. \tag{6.9}$$

Then the second term in the above expression becomes

$$\int \frac{dk}{(2\pi)^3} e^{-ik(x-x')} n_k = \frac{N}{V} e^{-\Delta^2(x-x')^2/4} e^{-ik_0(x-x')}, \tag{6.10}$$

up to a small error originating from the discrete integration, and for the third term

$$\begin{aligned} \frac{1}{V} \int \frac{dk}{(2\pi)^3} n_k^3 &= \frac{1}{V} \left(\frac{(2\pi)^3 N}{V(\sqrt{\pi}\Delta)^3} \right)^2 \int \frac{dk}{(2\pi)^3} e^{-2(k-k_0)^2/\Delta^2} \\ &\approx \frac{N^2}{V^3 \Delta^3}. \end{aligned} \tag{6.11}$$

Holding the density N/V and the width of the distribution Δ fixed, then the third term vanishes in the limit of large volumes V in (6.6) as $1/V$. The pair distribution function then becomes

$$\frac{N^2}{V^2} g(x-x') = \frac{N^2}{V^2} \left(1 + e^{-\Delta^2(x-x')/2} \right). \tag{6.12}$$

The probability to find bosons nearby, closer than a distance Δ^{-1}, is increased. This is an interesting phenomenon. Because of the symmetry alone, but not due to genuine interactions, bosons tend to stick together, to cluster or to “bunch”. The probability to find two bosons at exactly the same position is precisely twice as large as for large distances. This is no contradiction to what has been said before. The density can very well be constant, and at the same time the pair correlation functions indicates a clustering.

6.2 Photon correlations

Photons constitute the prototypical example of non-interacting bosons. They are indeed non-interacting to an extraordinarily good approximation. Making use of methods of quantum optics, one can prepare sophisticated states of light modes. One can also measure photon correlations in Hanbury-Brown Twiss experiments, as we briefly discuss in the lecture, but not in this script.

6.3 Weakly interacting dilute Bose gases

6.3.1 Quantum fluids and Bose Einstein condensation

Quantum degenerate Bose gases are physical systems that are still enjoying enormous attention in present-day research. An important bosonic fluid is He^4, exhibiting $S = 0$, a kind of Bose gas that features little interaction compared to other Bose gases made from heavier atoms. He^4 is fluid down to $T = 0$, turning to a superflud state at $T = 2.18K$ we come to later. The interaction of such Helium atoms is indeed small, but non-negligible. It can be well described by a *Lennard-Jones potential*,

$$V(r) = 4\varepsilon\left(\left(\frac{\sigma}{r}\right)^{12} - \left(\frac{\sigma}{r}\right)^{6}\right), \tag{6.13}$$

where $\varepsilon = 1.411 \times 10^{-15}$erg und $\sigma = 2.556 \times 10^{-10}$m. It captures a strong repulsion at short distances and a mild attraction at large distances.

In relatively recent years, *Bose Einstein condensation* has moved into the focus of significant attention both in experimental as well as in theoretical work, starting with the successful demonstration of Bose-Einstein-condensation of about 2000 ^{87}Rb atoms, followed by an experiment with 100000 ^{7}Li atoms and several millions ^{23}Na atoms. Today, the field of ultra-cold atomic quantum gases is one of the fastest growing research fields, giving rise to a number of exciting applications. We will later turn to a specific one, that of ultra-cold atoms in optical lattices, giving rise to instances of highly controlled quantum simulators. Before that, we will discuss in great detail the situation without the presence of an optical lattice, in form of the Bogoliubov-theory of weakly interacting Bose gases.

6.3.2 Bogoliubov theory of weakly interacting Bose gases

In the momentum representation we can write the Hamiltonian of interacting bosons as

$$H = \sum_k \frac{k^2}{2M} b_k^\dagger b_k + \frac{1}{2V}\sum_{k,p,q} V_q b_{k+q}^\dagger b_{p-q}^\dagger b_p b_k. \tag{6.14}$$

In the following, we will make use of a series of approximations that are well justified for dilute, weakly interacting Bose gases and give rise to a good model. To start with, V_q is the Fourier transform of the interaction in the position representation

$$V_q = \int dx e^{-iqx} V(x). \tag{6.15}$$

For low temperatures, Bose Einstein condensation takes place into the lowest mode corresponding to $k = 0$ statt. Indeed, at zero temperature and in the absence of interactions, so in case of the ideal Bose gas, this is the obvious ground state: Then the mode corresponding to $k = 0$ is simply occupied N times. The bosons “condense” in the same mode, which is perfectly possible for bosons, in contrast to the fermionic situation.

In analogy to the ideal Bose gas one should also expect that even for small interactions, this mode is macroscopically occupied. By this we mean that

$$N_0 = \langle \psi_N | a_0^\dagger a_0 | \psi_N \rangle \approx N \tag{6.16}$$

us expected to hold true. The number of particles outside this mode, so outside the condensate, is hence given by

$$N - N_0 \ll N_0 \sim N. \tag{6.17}$$

This leads us to the first approximation step: One can neglect all interactions outside the $k = 0$-mode, simply as densities are too small. We can hence discuss interactions exclusively of the $k = 0$ mode with those outside of it. Under these – indeed very mild – assumptions, we can write the Hamiltonian as

$$\begin{aligned} H &= \sum_k \frac{k^2}{2M} b_k^\dagger b_k + \frac{1}{2V} V_0 b_0^\dagger b_0^\dagger b_0 b_0 + \frac{1}{V} \sum_k{}' (V_0 + V_k) b_0^\dagger b_0 b_k^\dagger b_k \\ &+ \frac{1}{2V} \sum_k{}' V_k \left(b_k^\dagger b_{-k}^\dagger b_0 b_0 + b_0^\dagger b_0^\dagger b_k b_{-k} \right), \end{aligned} \tag{6.18}$$

in a form

- in which we have neglected polynomials higher than third degree in b_k.

Here V_0 again refers to the Fourier transform of the interaction (and not the volume). The symbol $\sum_k'$ again referes to a sum that takes out the term $k = 0$. Of course, we have that

$$b_0 | N_0, \dots \rangle = \sqrt{N_0} | N_0 - 1, \dots \rangle, \tag{6.19}$$

$$b_0^\dagger | N_0, \dots \rangle = \sqrt{N_0 + 1} | N_0 - 1, \dots \rangle. \tag{6.20}$$

Now the following approximations are plausible. To highlight them, they are all emphasized in an itemized environment in this text.

- Since Since N_0 is a large number, $N_0 \sim 10^{23}$, both terms largely correspond to the multiplication with the real number $\sqrt{N_0}$.
- What is more, in comparison to N_0, the impact of the commutator $[b_0, b_0^\dagger] = 1$ is negligible, and hence the operators

  $$b_0 = b_0^\dagger = \sqrt{N_0} \in \mathbb{R} \tag{6.21}$$

  can be replaced by real numbers.

This last step, needless to say, only makes sense for the mode $k = 0$. In this approximation, the Hamiltonian becomes

$$\begin{aligned} H &= \sum_k{}' \frac{k^2}{2M} b_k^\dagger b_k + \frac{1}{2V} N_0^2 V_0 \\ &+ \frac{N_0}{V} \sum_k{}' \left((V_0 + V_k) b_k^\dagger b_k + \frac{1}{2} V_k (b_k^\dagger b_{-k}^\dagger + b_k b_{-k}) \right). \end{aligned} \tag{6.22}$$

Interestingly, we do not know the precise value of N_0; we rather made some assumptions about it. It is determined by the N/V and implicitly by the interaction. We will now write total particle number operator of the entire system as

$$N_0 + \sum_k{}' b_k^\dagger b_k, \tag{6.23}$$

where were know that

$$\langle \psi_N | N_0 + \sum_k{}' b_k^\dagger b_k | \psi_N \rangle = N. \tag{6.24}$$

Hence, (6.23) can always be replaced by N. That is to say, by expanding this expression, we get terms of the form

$$\begin{aligned} \frac{V_0}{2V} N_0^2 &= \frac{V_0}{2V} N^2 - \frac{N V_0}{V} \sum_k{}' b_k^\dagger b_k \\ &+ \frac{V_0}{2V} \sum_{k,k'}{}' b_k^\dagger b_k b_{k'}^\dagger b_{k'}. \end{aligned} \tag{6.25}$$

In this way, the Hamiltonian becomes

$$\begin{aligned} H &= \sum_k{}' \frac{k^2}{2M} b_k^\dagger b_k + \frac{N}{V} \sum_k{}' V_k b_k^\dagger b_k \\ &+ \frac{N^2}{2V} V_0 \\ &+ \frac{N}{2V} \sum_k{}' V_k \left(b_k^\dagger b_{-k}^\dagger + b_k b_{-k} \right) + H', \end{aligned} \tag{6.26}$$

where

- H' is again a Hamiltonian with more than four creation or annihilation operators. Such terms are, however, linear in $(n')^2$, where

  $$n' = (N - N_0)/V \tag{6.27}$$

  is the density of those particles not contained in the condensate.

The Bogoliubov theory consists of this scheme of approximations. Let us briefly review this approximation scheme: It was key to the idea to single out the mode corresponding to $k = 0$ and to replace the respective operators by numbers (which makes perfect sense for large occupation numbers). Then we have neglected higher contributions to interaction terms, which is a good approximation for small densities. Indeed, when

$$n' \ll N/V \tag{6.28}$$

this is a good approximation, and in fact a very good approximation for weakly interacting, highly dilute Bose gases. Since we have neglected H', we have

$$\begin{aligned} H &= {\sum_k}' \frac{k^2}{2M} b_k^\dagger b_k + \frac{N}{V} {\sum_k}' V_k b_k^\dagger b_k \\ &+ \frac{N^2}{2V} V_0 \\ &+ \frac{N}{2V} {\sum_k}' V_k \left(b_k^\dagger b_{-k}^\dagger + b_k b_{-k} \right). \end{aligned} \tag{6.29}$$

This is again a quadratic expression in bosonic operators in the Hamiltonian. Such systems are called *free systems*, or *non-interacting systems*, or *quasi-free systems*, the latter in particular in the mathematical literature. Such systems for which the polynomial is a polynomial of second order in bosonic or fermionic operators are exactly solvable. Indeed, they constitute some of the few solvable systems in physics. The mindset here is to understand in what way one can make approximation schemes, until one arrives at such a quadratic Hamiltonian problem. We will solve this model first, and later have a more systematic look at such problems.

Let us now make use of the transformation

$$b_k = u_k a_k + v_k a_{-k}^\dagger, \tag{6.30}$$

$$b_k^\dagger = u_k a_k^\dagger + v_k a_{-k}, \tag{6.31}$$

where $u_k, v_k \in \mathbb{R}$, from one set of valid bosonic annihilation operators $\{b_k\}$ to a new one $\{a_k\}$. This is only a legitimate transformation to such a new set if the new operators again satisfy

$$[a_k, a_{k'}] = 0, \tag{6.32}$$

$$[a_k^\dagger, a_{k'}^\dagger] = 0, \tag{6.33}$$

$$[a_k, a_{k'}^\dagger] = \delta_{k,k'}, \tag{6.34}$$

which is nothing but[1]

$$u_k^2 - v_k^2 = 1 \tag{6.37}$$

for all k. The inverse transformation is then given by

$$a_k = u_k b_k - v_k b_{-k}^\dagger \tag{6.38}$$

as one easily finds. Such a transformation is a *symplectic transformation*. Note that it is not a linear transformation of creation operators, but a mixed transformation that brings

[1] The proof is rather obvious, as

$$[a_k, a_{k'}] = u_k v_{k'} \delta_{k,-k'} + v_k u_{k'} (-\delta_{k,-k'}) = 0, \tag{6.35}$$

and similarly

$$[a_k, a_{k'}^\dagger] = u_k u_{k'} \delta_{k,k'} + v_k v_{k'} (-\delta_{k,k'}) = (u_k^2 - v_k^2) \delta_{k,k'}. \tag{6.36}$$

together annihilation and creation operators. We will later identify the transformation implemented here as a symplectic transformation.

Making use of a transformation that transforms the previous particles into "quasi-particles", we hence have

$$\begin{aligned} b_k^\dagger b_k &= u_k^2 a_k^\dagger a_k + v_k^2 a_{-k} a_{-k}^\dagger + u_k v_k (a_k^\dagger a_{-k}^\dagger + a_k a_{-k}), && (6.39)\\ b_k^\dagger b_{-k}^\dagger &= u_k^2 a_k^\dagger a_{-k}^\dagger + v_k^2 a_k a_{-k} + u_k v_k (a_k^\dagger a_k + a_{-k} a_{-k}^\dagger), && (6.40)\\ b_k b_{-k} &= u_k^2 a_k a_{-k} + v_k^2 a_k^\dagger a_{-k}^\dagger + u_k v_k (a_{-k}^\dagger a_{-k} + a_k a_k^\dagger). && (6.41) \end{aligned}$$

In these coordinates, the Hamiltonian is given by

$$\begin{aligned} H &= \frac{1}{2V} N^2 V_0 && (6.42)\\ &+ \sum_k{}' \left(\frac{k^2}{2M} + \frac{N}{V} V_k\right) \left(u_k^2 a_k^\dagger a_k + v_k^2 a_k a_k^\dagger + u_k v_k (a_k^\dagger a_{-k}^\dagger + a_k a_{-k})\right) \\ &+ \frac{N}{2V} \sum_k{}' V_k \left((u_k^2 + v_k^2)\left(a_k^\dagger a_{-k}^\dagger + a_k a_{-k}\right) + 2 u_k v_k (a_k^\dagger a_k + a_k a_k^\dagger)\right). \end{aligned}$$

We have not yet made of all the freedom we have, however: We still have the freedom to let the non-diagonal terms vanish, without approximation. This means that (i) for all k

$$\left(\frac{k^2}{2M} + \frac{N}{V} V_k\right) u_k v_k + \frac{N}{2V} V_k (u_k^2 + v_k^2) = 0 \tag{6.43}$$

should hold. Together with (ii) $u_k^2 - v_k^2 = 1$ for all k from the preservation of the commutation relations, we hence have a system of equations in $\{u_k\}$ und $\{v_k\}$. This allows us to identify the coefficients $\{u_k\}$ and $\{v_k\}$. It is helpful to introduce the following quantities talking – maybe unsurprisingly – the role of frequencies,

$$\begin{aligned} \omega_k &= \left(\left(\frac{k^2}{2M} + \frac{N}{V} V_k\right)^2 - (N V_k / V)^2\right)^{1/2} \\ &= \left(\left(\frac{k^2}{2M}\right)^2 + \frac{N k^2 V_k}{V M}\right)^{1/2}. && (6.44) \end{aligned}$$

In this way, we find the solution for the coefficients $\{u_k\}$ and $\{v_k\}$ as

$$\begin{aligned} u_k^2 &= \frac{\omega_k + \left(\frac{k^2}{2M} + \frac{N}{V} V_k\right)}{2\omega_k}, && (6.45)\\ v_k^2 &= \frac{-\omega_k + \left(\frac{k^2}{2M} + \frac{N}{V} V_k\right)}{2\omega_k}, && (6.46)\\ u_k v_k &= -\frac{N V_k}{2 V \omega_k}. && (6.47) \end{aligned}$$

This finally gives for the Hamiltonian the following expression. Before spelling it out, let us remind outselves that the last steps have been nothing but a way to decouple a quadratic systems by means of a clever choice of coordinates. Such an approach is always possible for quadratic systems, as we will see later. We therefore arrive at the following simple Hamiltonian.

Hamiltonian of the weakly interacting Bose gas in the Bogoliubov approximation:

$$H = \frac{N^2}{2V}V_0 - \frac{1}{2}\sum_k{}' \left(\frac{k^2}{2M} + \frac{N}{V}V_k - \omega_k\right) + \sum_k{}' \omega_k a_k^\dagger a_k. \quad (6.48)$$

We can easily interpret this Hamiltonian. The first term is a number, the ground state energy. The second term is a collection of harmonic oscillators of different frequency, reflecting excitations. These excitations, generated by $a_k^\dagger$, are also called "quasi-particles", in fact quasi-particles with wave number k. These creation operators correspond to both creation and annihilation operators in the original coordinates, it is important to emphasize.

The ground state of the systems with N particles is the vacuum with state vector $|\o\rangle$, in these coordinates. Then we have

$$a_k|\o\rangle = 0 \quad (6.49)$$

for all k. In this ground states, that is, at zero temperature in a language of statistical physics, we can now determine the number of particles (not quasi-particles), referring to the original bosonic operators $\{b_k\}$. This number is given by

$$\begin{aligned} N' &= \langle\o|\sum_k{}' b_k^\dagger b_k|\o\rangle = \langle\o|\sum_k{}' v_k^2 a_k a_k^\dagger|\o\rangle \\ &= \sum_k{}' v_k^2, \end{aligned} \quad (6.50)$$

where we have made use of the connection between quasi-particles and particles. That means: Without any interaction, all particles are condensed. When taking interactions into account, it is still true that no quasi-particles are excited in the ground state. But this now means that some original particles are outside the condensate.

We would now like to make another assumption, that is,

- about the specific interaction of the particles:

We make the plausible assumptions that bosons are weakly interaction via a *contact potential* such as hard balls. This means that the potential is given by

$$V(x) = \lambda\delta(x). \quad (6.51)$$

Making use of (6.44) as well as (6.45) and (6.47), we find for the number density of the particles outside the condensates[2]

$$n' = \frac{N'}{V} = \frac{M^{3/2}}{3\pi^2}(\lambda N/V)^{3/2}. \tag{6.52}$$

The number of particles in the condensate is then

$$N_0 = N - N' = N - n'V. \tag{6.53}$$

This is no longer N, but reduced due to the interaction. Equipped with these insights, we can also revisit the ground state energy: In (6.48) it consists of a term that would be the interaction energy, were all particles contained in the condensate, and another negative term. Due to the occupation of bosonic states with $k \neq 0$ in the ground state the kinetic energy is increased, while the potential energy is reduced.

6.3.3 Dispersion relation in the Bogoliubov theory

As mentioned before, the elementary excited states are those for which one applies $a_k^\dagger$ to $|\o\rangle$: This gives quasi-particles with wave number k. Their energy is, as being determined by the Hamiltonian, simply ω_k. For small k we find that ω_k is according to (6.44) essentially linear in k, and one gets

$$\omega_k \approx ck, \tag{6.54}$$

where

$$c = \left(\frac{NV_0}{VM}\right)^{1/2}. \tag{6.55}$$

Excitations with long wave length are hence those with linear dispersion. c can be seen as the *speed of sound* that determines the velocity with which excitations travel, a quantity that can be related to the actual information propagation speed. For large k one finds according to (6.44)

$$\omega_k \approx \frac{k^2}{2M} + \frac{N}{V}V_k, \tag{6.56}$$

a quadratic expression. This is the dispersion relation of free particles, the energy of which is shifted by an average potential of NV_k/V. We will now assume – stretching the above assumptions to an extent – that the interaction is sufficiently strong so that we have a local minimum of ω_k as a function of k that is different from $k = 0$: The function increases linearly, the decreases again, only to grow subsequently. Such a behaviour is explained by the above dispersion relation.[3]

[2]Here we encounter the subtlety that the density is not analytic in λ and simple perturbation theory does not work.

[3]Let us not be too pedantic here and be too worried about whether for such strong interactions the Bogoliubov theory may no longer deliver good approximations. More exact computations indeed confirm this feature of the dispersion relation that we will accept as the proper dispersion relation of the type of system at hand.

6.3.4 Superfluidity

We have just seen that the dispersion relation of the quasi-particles of the weakly interacting dilute Bose gas first grows linearly and for large k grows quadratically. What is more, we find a local minimum of ω_k as a function of k, different from zero. We will now see that such a dispersion relation has a remarkable implication: Such systems can be superfluid. *Superfluidity* is a remarkable phenomenon: They are fluids without any viscosity. They flow around objects, and items can be dragged through such fluids without losing velocity. It is a perfectly frictionless fluid.

This behaviour is a consequence of the dispersion relation, for which

$$in_k \frac{\omega_k}{k} = v_{\text{Crit}} > 0 \tag{6.57}$$

holds true. Again, for small values of k the dispersion relation is linear. Such quasi-particles are called *phonons*. Excitations with a k nearby the local minimum are called, for historical reasons, *rotons*.[4] We will now see how this dispersion relation gives rise to superfluidity. In order to see this, we will investigate a tube through which a superfluid is flowing. We will look at this system in two reference frames:

- In the first reference frame B, the tube is at rest and the fluid moves with velocity $-v$.
- In the other reference frame B', the fluid is at rest, and the tube moves with velocity v.

Of course, both are perfectly legitimate references frame, related to each other by a *Galilei transformation*.[5]

[4] For superfluid He^4 this minimum is at $k_0 = 1.91 \times 10^{10} m\hbar$. The effective mass of such rotons is about 0.16 the mass of Helium, and they have an energy gap of $\Delta/k = 8.6$K.

[5] Here a brief reminder on Galilei transformations: Let us assume we have N Teilchen, the above velocity is v and the particles have positions $\{x_j\}$ and momenta $\{p_j\}$. Then we have in the above reference frames

$$\begin{aligned} x_j &= x'_j - vt, \qquad (6.58)\\ p_j &= p'_j - Mv. \qquad (6.59)\end{aligned}$$

This means that

$$P = \sum_j p_j = \sum_j (p'_j - mv) = p' - Mv. \tag{6.60}$$

The energy transforms as (note that V is here the interaction, not the velocity)

$$\begin{aligned} E &= \sum_j \frac{p_j^2}{2M} + \sum_{\langle j,k\rangle} V(x_j - x_k)\\ &= \sum_j \frac{M}{2}\left(\frac{p'_j}{M} - v\right)^2 + \sum_{\langle j,k\rangle} V(x'_j - x'_k)\\ &= \sum_j \frac{(p'_j)^2}{M} - p'v + \frac{1}{2}Mv^2 + \sum_{\langle j,k\rangle} V(x'_j - x'_k)\\ &= E' - p'v + \frac{1}{2}Mv^2. \end{aligned} \tag{6.61}$$

This is precisely the transformation that we need, and it makes little difference whether the particles are classical or quantum.

We now allow for a little bit more, namely that in B' the fluid has energy E' and momentum P'. The energy E in reference frame B and the momentum are given by

$$\begin{aligned} P &= P' - Mv, && (6.62)\\ E &= E' - Pv + \frac{1}{2}Mv^2. && (6.63) \end{aligned}$$

Let us discuss why there is no friction up to a *critical velocity*. Friction means that quasi-particles are being generated that dissipate energy and transfer this energy to other, undirected degrees of freedom. If there are no quasi-particles, there is no friction and dissipation. Viewed in reference frame B', when encountering dissipation quasi-particles will have to be generated that move with the tube. In the reference frame of the tube it looks as if the fluid was decelerated.

Of course, such excitations will only arise if they are energetically favoured, and this insight is at the heart of the matter. Let us begin in the ground state at temperature $T = 0$. In the reference frame B', energy and momentum are given by

$$\begin{aligned} E' &= E'_g, && (6.64)\\ p' &= 0. && (6.65) \end{aligned}$$

In B, these expressions are

$$\begin{aligned} E_g &= E'_g + \frac{1}{2}Mv^2 && (6.66)\\ p &= -Mv. && (6.67) \end{aligned}$$

If a quai-particle with momentum $p = k$ (actually, $p = \hbar k$, but we have set $\hbar = 1$) and energy ω_k is being generated, energy and momentum in B' are given by

$$\begin{aligned} E' &= E'_g + \omega_k,\\ p' &= k, && (6.68) \end{aligned}$$

and hence in B'

$$\begin{aligned} E' &= E'_g + \omega_k - kv + \frac{1}{2}Mv^2,\\ p' &= k - Mv. && (6.69) \end{aligned}$$

The excitation energy in B equals, again with $\hbar = 1$,

$$\Delta E = \omega_k - kv. \qquad (6.70)$$

For this reason, ΔE is the change of energy of the fluid due to the generation of a quasi-particle in reference frame B. Only if

$$\Delta E < 0 \qquad (6.71)$$

the fluid will lessen its energy by the dissipation process. Of course, this means that, seeing the problem as a one-dimensional problem, that

$$v > \frac{\omega_k}{k} \qquad (6.72)$$

has to hold, such that energy is lost in the first place, for this excitation labeled k. In other words, the velocity has to be larger than a critical velocity.

What does the dispersion relation have to do with all this? Well, everything. The thing is that if (6.57) holds true and a critical velocity $v_{\rm Crit}$ exists, then there is no k for

$$v < v_{\rm Crit} \tag{6.73}$$

that satisfies (6.72). Therefore, there cannot be excitations that are energetically favoured, and hence there is no dissipation. The phenomenon of superfluidity is hence a consequence of a curious dispersion relation, in which the energy per wave number is bounded from below. We have – within a framework of second quantization – hence understood how quantum gases can flow entirely without friction and dissipation as a genuine quantum effect. This is one of the most curious and intriguing macroscopic quantum phenomena. For fluid Helium, this effect occurs at temperatures $T < 2.18$K and can well be observed. This is one of the core results of this course.

6.3.5 Phenomenology of superconductivity

Until now the discussion was limited to the situation of zero temperature $T = 0$. As mentioned before, the temperature does not have to be strictly zero (a fictitious state of affairs anyway) in order to arrive at a superfluid. For temperatures different from zero, superfluids are captured rather well by the *two fluid model* due to Tizsa: It is a purely phenomenological model that regards such superfluid as being composite of a superfluid component (the actual superfluid) and a normal component (that for low temperatures largely consists of phonons). The superfluid component is assumed to have

- zero entropy and
- can move freely through capillaries without any friction.

The normal component has a non-zero entropy, and also a viscosity different from zero. Based on this crude model, a number of effects can be nicely explained.

- For example, the effect can be explained in which a small hole in a tank filled with He II can serve as a "filter" for the superfluid component. If one connects two tanks via a thin capillary and puts some pressure onto one tank, the fluid will flow from one tank A to the other B. But then, because the superfluid component has no entropy, the entropy per mass in tank A will increae and that in B will decrease. Hence, A will cool down, while B will heat up. This is indeed observed, in a fascinating effect called *mechanocaloric effect*.
- A further quite spectacular effect is called *fountain effect*: It is basically the opposite effect in which a temperature gradient gives rise to a difference in pressure. If one does it well, one can generated nice fountains, hence the name.
- Finally, there is an effect called *second sound*: If one generates a sound wave, both the normal and the superfluid component will usually oscillate, and hence

one will arrive at a sinusoidal modulation of the mass density. But now there is also another type of oscillation conceivable in which both components oscillate against each other with an opposite phase. This would not be a sound wave in the ordinary sense, as the mass density will remain constant in time. But it would give rise to a modulation of the entropy density, for the above mentioned reason. One could therefore hope that by means of clever local heating one can generate such waves. Then the temperature gradient would not diffusively propagate, but ballistically, like a wave, propagating with some speed of sound. This could be called a sound wave of second sound, and indeed, it can be observed in experiments.