# 2. Pauli matrices and Bloch sphere

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

a) first observation: every Pauli matrix squares to $\mathbb{1}$:

$$X^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}$$

$$Y^2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = \begin{pmatrix} i(-i) & 0 \\ 0 & (-i)i \end{pmatrix} = \mathbb{1}$$

$$Z^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}$$

We can use this for:

- $XZX = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix} = -Z$

$\Longleftrightarrow \boxed{XZ = -ZX}$

similarly, for other pairs of Pauli matrices:

- $XYX = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = -Y \Longleftrightarrow \boxed{XY = -YX}$

- $ZYZ = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = -Y$

$\Longleftrightarrow \boxed{ZY = -YZ}$

b) explicit calculations give:

$$X \otimes X = \begin{pmatrix} 0 & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & 0 \end{pmatrix}$$

$$Z \otimes Z = \begin{pmatrix} 1 & & & \\ & -1 & & 0 \\ 0 & & -1 & \\ & & & 1 \end{pmatrix}$$

$$X \otimes Y = \begin{pmatrix} 0 & & & i \\ & & -i & \\ & i & & 0 \\ -i & & & \end{pmatrix}$$

$$Y \otimes X = \begin{pmatrix} 0 & & & i \\ & & i & \\ & -i & & 0 \\ -i & & & \end{pmatrix}$$

note: $\neq X \otimes Y$

c) $XZ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -iY$

<u>note</u>: it holds for any pair of Pauli matrices $(\sigma_i, \sigma_j)$, that $\sigma_i \cdot \sigma_j \sim \sigma$ where $\sigma$ is (another) Pauli matrix. In this sense, they are <u>closed</u> under multiplication.

d) dimension of $M(\mathbb{C}^2)$ is 4 ($2 \times 2$ matrices). Hence, we need 4 basis elements $\{\mathbb{1}, X, Y, Z\} =: \mathcal{B}$

<u>to show</u>: $\langle b_i, b_j \rangle = 0$ for $i \neq j$ (orthogonality)

1. case $b_i = \mathbb{1}$, $b_j = \sigma$ Pauli matrix

$$\langle \mathbb{1}, \sigma \rangle = Tr(\sigma) = 0$$

no "dagger" needed since $\sigma^\dagger = \sigma \ \forall \sigma \in \mathcal{B}$

2. case $b_i = \sigma_i, b_j = \sigma_j \neq \sigma_i$

some other Pauli matrix, see c)

$$\langle \sigma_i, \sigma_j \rangle = Tr(\sigma_i \sigma_j) \sim Tr(\sigma) = 0$$

by 1. case

$\Rightarrow$ elements in $\mathcal{B}$ are pairwise orthogonal & form basis of $h(\mathbb{C}^2)$

e)

Frobenius norm $\langle b_i, b_i \rangle = Tr(b_i^2) = Tr(\mathbb{1}) = 2$

$b_i = \mathbb{1} \rightarrow$ trivially true

$b_i \in \{X, Y, Z\}$ see a), each squares to $\mathbb{1}$

$$\Rightarrow \boxed{\text{normalized } \tilde{\mathcal{B}} = \{\tfrac{1}{\sqrt{2}}\mathbb{1}, \tfrac{1}{\sqrt{2}}X, \tfrac{1}{\sqrt{2}}Y, \tfrac{1}{\sqrt{2}}Z\}}$$

f) density matrix is $\rho \in h(\mathbb{C}^2)$

as shown any element in $h(\mathbb{C}^2)$ can be represented as

$$\rho = a\mathbb{1} + bX + cY + dZ \quad ; \ a, b, c, d \in \mathbb{R}$$

as density matrix, however, has the following properties:

1. $tr(\rho) = 1$

2. $tr(\rho^2) \leq 1$

3. non-negative, i.e. only positive eigenvalues

from 1. it follows:

$$1 = \mathrm{tr}(\rho) = a\,\underbrace{\mathrm{tr}(\mathbb{1})}_{2} + b\,\underbrace{\mathrm{tr}(X)}_{0} + c\,\underbrace{\mathrm{tr}(Y)}_{0} + d\,\underbrace{\mathrm{tr}(Z)}_{0}$$

$$= 2a \quad \Longleftrightarrow \quad a = \frac{1}{2}$$

for convenience, let us define $b = \frac{1}{2}\hat{b}$, $c = \frac{1}{2}\hat{c}$, $d = \frac{1}{2}\hat{d}$:

$$\rho = \frac{1}{2}\left(\mathbb{1} + \hat{a}X + \hat{b}Y + \hat{c}Z\right)$$

$$= \frac{1}{2}\left(\mathbb{1} + \vec{a}\cdot\vec{\sigma}\right)$$

$$\begin{pmatrix}\hat{a}\\ \hat{b}\\ \hat{c}\end{pmatrix}\cdot\begin{pmatrix}X\\ Y\\ Z\end{pmatrix}$$

note: $\mathrm{Tr}\left((\vec{a}\cdot\vec{\sigma})(\vec{a}\cdot\vec{\sigma})\right) = 2|\vec{a}|^2$

by orthogonality of $\{X, Y, Z\}$

$$3. \Rightarrow 1 \geq \mathrm{tr}(\rho^2) = \frac{1}{4}\mathrm{Tr}\left(\mathbb{1} + 2\,\vec{a}\cdot\vec{\sigma} + (\vec{a}\cdot\vec{\sigma})^2\right)$$

$$= \frac{1}{4}\left(2 + 0 + 2|\vec{a}|^2\right)$$

$$= \frac{1}{2}\left(1 + |\vec{a}|^2\right)$$

$$\Longleftrightarrow \quad 2 \geq 1 + |\vec{a}|^2 \quad \Longleftrightarrow \quad \boxed{1 \geq |\vec{a}|^2 = \hat{a}^2 + \hat{b}^2 + \hat{c}^2}$$

note: alternatively, one could show that

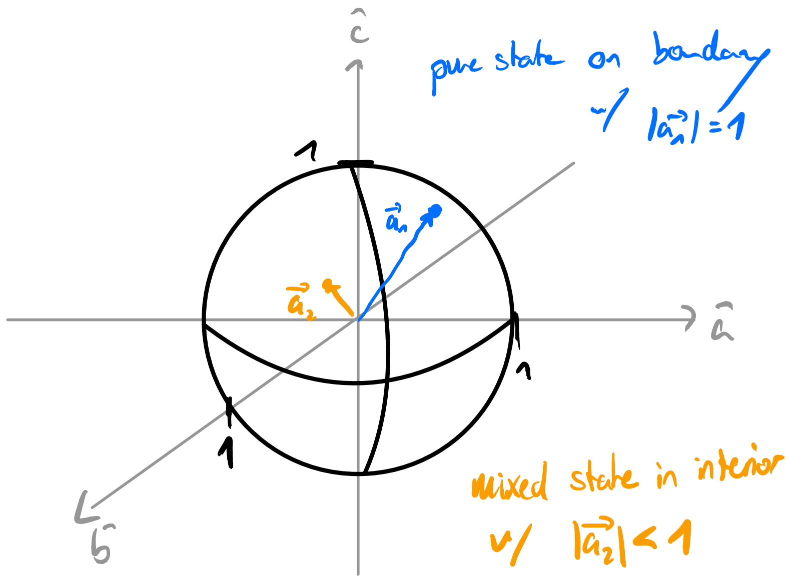
$\rho$ has eigenvalues $\frac{1}{2}(1 \pm |\vec{a}|)$ which have to be

positive by 3. $\Rightarrow |\vec{a}| \leq 1$.

g)

pure states fulfill $tr(\rho^2) = 1$, i.e saturate the inequality in f).

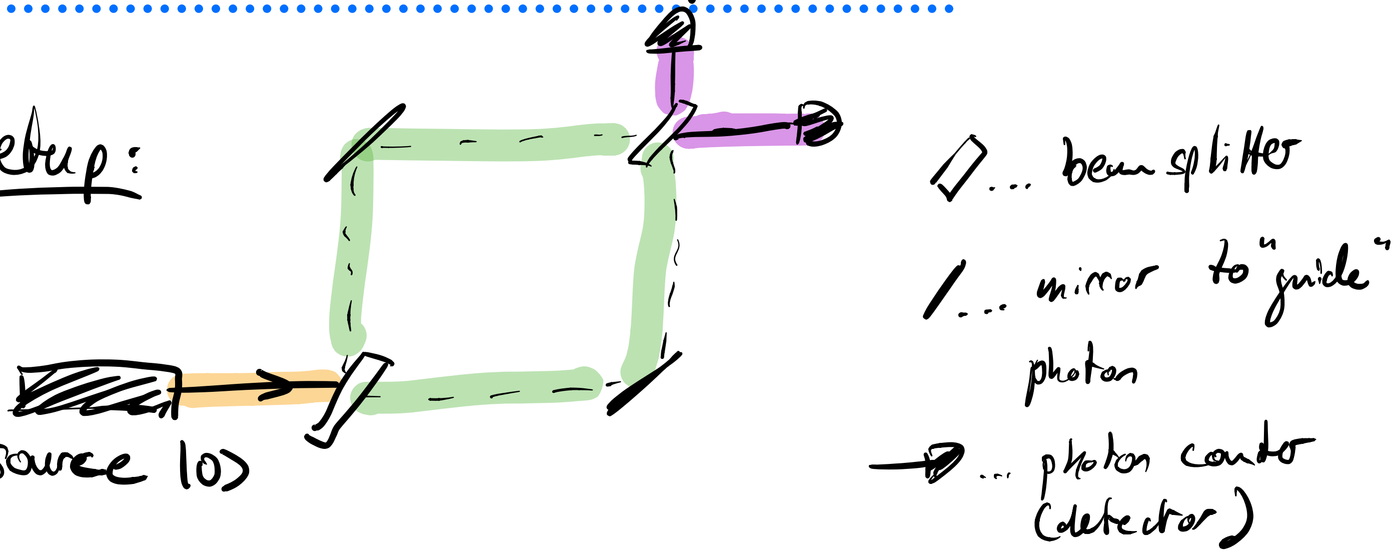
$\Rightarrow$ Fulfill $|\vec{a}| = 1$, meaning they live on the boundary of the Bloch ball




any point within $\oplus$ corresponds to a physical qubit state

# 3. Beam splitters and interferometers

setup:




a) note: $S' = S$, i.e. reflection and transmission prob. $P_S(R)$ and $P_S(T)$ is the same for each basis state $\{|0\rangle, |1\rangle\}$

We calculate:

$$P_S(R) = |\langle 1|S|0\rangle|^2 = \frac{1}{2} \qquad P_S(T) = |\langle 0|S|0\rangle|^2 = \frac{1}{2}$$

both reflection and transmission prob. are 50%.

b) state in region: $|0\rangle$

— " — region: $S|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$

— " — — " —: $S'S|0\rangle = S^2|0\rangle$

$$= iX|0\rangle = i|1\rangle$$

$$S^2 = \frac{1}{2}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix} = \dots = \begin{pmatrix}0 & i\\ i & 0\end{pmatrix} = iX$$

$\Rightarrow$ when input state is $|0\rangle$, after 2 beam splitters, we observe $|1\rangle$ with 100% certainty (and $|0\rangle$ with 0%).

c) Prob after $N$-beamsplitters $P_{S^N}(|i\rangle) = p_i \quad i=0,1$

$N = 1 \quad 2 \quad 3 \quad 4 \ldots$

as calculated: $S^2 = iX \Rightarrow S^4 \sim \mathbb{1}$ meaning,

4 beamsplitters <u>do nothing</u>.

hence, only 4 different probability distributions have to be considered:

1. $N = 0 \mod 4$:

$$p_0 = P_{S^0}(|0\rangle) = |\langle 0|S^0|0\rangle|^2 = |\langle 0|\mathbb{1}|0\rangle|^2 = 1$$
$$p_1 = P_{S^0}(|1\rangle) = |\langle 1|S^0|0\rangle|^2 = \ldots = 0$$

2. $N = 1 \mod 4$:

$$p_0 = P_S(|0\rangle) = |\langle 0|S|0\rangle|^2 = \frac{1}{2}$$
$$p_1 = P_S(|1\rangle) = |\langle 1|S|0\rangle|^2 = \frac{1}{2}$$

from a)

3. $N = 2 \mod 4$

$$p_0 = P_{S^2}(|0\rangle) = |\langle 0|S^2|0\rangle|^2 = |\langle 0|iX|0\rangle|^2 = 0$$
$$p_1 = P_{S^2}(|1\rangle) = |\langle 1|S^2|0\rangle|^2 = 1$$

from b)

4. $N=3 \mod 4$

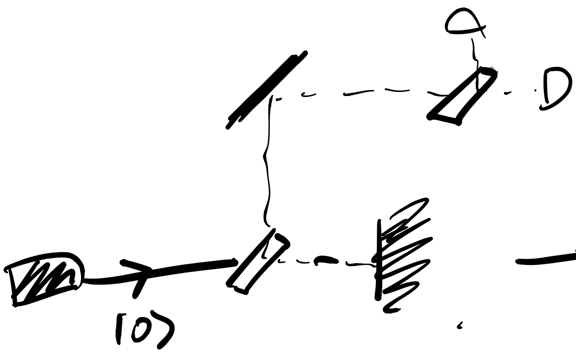
$$P_0 = P_{S^3}(|0\rangle) = |\langle 0| S^3 |0\rangle|^2 = |\langle 0| iXS|0\rangle|^2$$

$$= \frac{1}{2} |\langle 1| (|0\rangle + i|1\rangle)|^2 = \frac{1}{2}$$

$$P_1 = P_{S^3}(|1\rangle) = |\langle 1| S^3 |0\rangle|^2 = \ldots = \underline{\underline{\frac{1}{2}}}$$

These distributions repeat after 4 splitters.

d) block 1 path:




→ we "forget" about $|0\rangle$ state after first $S$

$\Rightarrow$ final state (unnormalized) $|\tilde{f}\rangle = S\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} S|0\rangle$

$$= S|1\rangle\underbrace{\langle 1|S|0\rangle}_{= \frac{i}{\sqrt{2}}}$$

$$= \frac{i}{\sqrt{2}} S|1\rangle$$

normalized final state $|f\rangle = S|1\rangle = \frac{1}{\sqrt{2}}(i|0\rangle + |1\rangle)$

$\Rightarrow$ output probability distribution is 50:50 again!

"blocking" is detected by upper photon <u>not</u> interfering with lower one at 2nd beam splitter.

e)

def.: $\tilde{S}(p)$ s.t. $P_{\tilde{S}(p)}(|0\rangle) = p \Rightarrow P_{\tilde{S}(p)}(|1\rangle) = 1-p$

conditions:

1. $\tilde{S}(p)^T = \tilde{S}(p)$

2. $\langle 0|\tilde{S}(p)|0\rangle = \langle 1|\tilde{S}(p)|1\rangle$

"symmetric"

$\Rightarrow$ 2 matrix elements open namely $\langle 0|S(p)|0\rangle =: \alpha$

$\langle 0|\tilde{S}(p)|1\rangle =: \beta$ $\quad \alpha, \beta \in \mathbb{C}$

$$\Rightarrow \quad \tilde{S}(p) = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}$$

transmission probability: $|\alpha|^2 \overset{!}{=} 1 \Rightarrow \alpha = c \cdot \sqrt{p}$ w/ $|c| = 1$

reflection —"— : $|\beta|^2 \overset{!}{=} 1-p \Rightarrow \beta = d\sqrt{1-p}$ w/ $|d| = 1$

moreover, $\tilde{S}$ has to be unitary:

$$\tilde{S}(p)^+ \overset{!}{=} \tilde{S}(p)^{-1} \Leftrightarrow \tilde{S}(p)\tilde{S}^+(p) = \mathbb{1}$$

calculate:

$$\tilde{S}(p)\,\tilde{S}(p)^+ = \begin{pmatrix} c\sqrt{p} & d\sqrt{1-p} \\ d\sqrt{1-p} & c\sqrt{p} \end{pmatrix} \begin{pmatrix} c^*\sqrt{p} & d^*\sqrt{1-p} \\ d^*\sqrt{1-p} & c^*\sqrt{p} \end{pmatrix}$$

$$= \begin{pmatrix} p|c|^2+(1-p)|d|^2 & (cd^* + c^* d)\sqrt{p}\sqrt{1-p} \\ (cd^* + c^* d)\sqrt{p}\sqrt{1-p} & p|c|^2+(1-p)|d|^2 \end{pmatrix}$$

$|c| = |d| = 1$, $cd^* + c^* d = 2\mathrm{Re}(c^* d)$

$$= \begin{pmatrix} 1 & \mathrm{Re}(c^* d)\sqrt{p(1-p)} \\ \mathrm{Re}(c^* d)\sqrt{p(1-p)} & 1 \end{pmatrix}$$

$$\overset{!}{=} \underline{\mathbb{1}}$$

$\Rightarrow$ $\mathrm{Re}(c^* d) = 0$, i.e. $c^* d$ has to be imaginary

if we choose $d = 1$, $c = \pm i$ and

$$\boxed{\hat{S}(p) = \begin{pmatrix} \sqrt{p} & \pm i\sqrt{1-p} \\ \pm i\sqrt{1-p} & \sqrt{p} \end{pmatrix}}$$

possible to have transmission & reflection probabilities dep. on input, i.e. have two propabilites $p_1, p_2$ ? Following first part of construction above and loosing 1. & 2. we get:

$$\hat{S}(p_1, p_2) = \begin{pmatrix} a\sqrt{p_1} & b\sqrt{1-p_2} \\ c\sqrt{1-p_1} & d\sqrt{p_2} \end{pmatrix}; \quad 1 = |a| = |b| = |c| = |d|$$

can $\hat{S}(p_1, p_2)$ be unitary ?

calculate:

$$\hat{S}(p_1,p_2)\hat{S}(p_1,p_2)^\dagger = \begin{pmatrix} a^*\sqrt{p_1} & c^*\sqrt{1-p_1} \\ b^*\sqrt{1-p_2} & d^*\sqrt{p_2} \end{pmatrix}\begin{pmatrix} a\sqrt{p_1} & b\sqrt{1-p_2} \\ c\sqrt{1-p_1} & d\sqrt{p_2} \end{pmatrix}$$

$$= \begin{pmatrix} |a|^2 p_1 + |c|^2(1-p_1) & a^*b\sqrt{p_1(1-p_2)} + c^*d\sqrt{1-p_1}\sqrt{p_2} \\ ab^*\sqrt{p_1}\sqrt{1-p_2} + d^*c\sqrt{p_2}\sqrt{1-p_1} & |b|^2(1-p_2) + |d|^2 p_2 \end{pmatrix}$$

$|a|,|b|,|c|,|d| = 1$

$$= \begin{pmatrix} 1 & a^*b\sqrt{p_1(1-p_2)} + c^*d\sqrt{p_2(1-p_1)} \\ \left[a^*b\sqrt{p_1(1-p_2)} + c^*d\sqrt{p_2(1-p_1)}\right]^* & 1 \end{pmatrix}$$

$$\overset{!}{=} \mathbb{1}$$

$$\Rightarrow a^*b\sqrt{p_1(1-p_2)} + c^*d\sqrt{p_2(1-p_1)} \overset{!}{=} 0$$

either: $a^*b = 0 \Rightarrow a = 0 \vee b = 0$ ϟ since $|a| = |b| = 1$

or: $p_1 = p_2 \Rightarrow a^*b + c^*d = 0$

$\Leftrightarrow a^*b = -c^*d$

has solution: for example $a = d = 1$ and

$b = 1, c = -1 \longrightarrow$ "hardamard gate" for $p = \frac{1}{2}$

$b = i, c = i \longrightarrow$ case a) for $p = \frac{1}{2}$

$\Rightarrow$ a unitary theory does not allow for $p_1 \neq p_2$ !

But for $\tilde{S}^T \neq \bar{S}$ (non-symmetric beam splitter)