Freie Universität Berlin
Tutorials on Quantum Information Theory
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Problem Sheet 3
Teleportation and Introduction to Graphical Calculus

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1. **Schmidt decomposition and purification**

In the lecture, you already saw the Schmidt decomposition of bipartite quantum states $|\Psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$ as given by

$$|\Psi\rangle = \sum_{i=1}^{d} \sqrt{\lambda_j}\, |\psi_j^1\rangle\, |\psi_j^2\rangle\,,$$

where $\{|\psi_j^i\rangle\}$ are orthonormal bases of $\mathcal{H}_i$.

In this exercise, we will study some useful properties and applications of the Schmidt decomposition. To begin with, let us look at states with the same Schmidt coefficients, that is

$$|\Psi\rangle = \sum_{i=1}^{d} \sqrt{\lambda_j}\, |\psi_j^1\rangle\, |\psi_j^2\rangle\,, \quad |\Phi\rangle = \sum_{i=1}^{d} \sqrt{\lambda_j}\, |\phi_j^1\rangle\, |\phi_j^2\rangle\,.$$

a) Show that $|\Psi\rangle$ and $|\Phi\rangle$ are related by a local unitary, i.e., a unitary of the form $U \otimes V$ with U and V unitary. Give that unitary explicitly.

b) Show that any local unitary transformation leaves the Schmidt coefficients invariant.

This gives rise to a nice interpretation of the Schmidt coefficients of a state in terms of entanglement (more soon!):

c) Determine the reduced density matrices $\rho_1 = \mathrm{Tr}_2\, |\Psi\rangle\langle\Psi|$ and $\rho_2 = \mathrm{Tr}_1\, |\Psi\rangle\langle\Psi|$. How can the Schmidt coefficients be interpreted? What are the Schmidt coefficients of the maximally entangled state?

d) Use the Schmidt decomposition to show that *any* bipartite state $|\Psi\rangle$ can be expressed as

$$|\Psi\rangle = (X \otimes \mathbb{1})\, |\Omega\rangle\,,$$

where $|\Omega\rangle$ is a maximally entangled state.

The maximally entangled state is *invariant* under certain product unitaries $U \otimes V$.

e) What are the conditions on U and V for this to be the case?

Recall from the lecture that for any quantum state $\rho \in \mathcal{L}(\mathcal{H})$ there exists a pure quantum state $|\psi_\rho\rangle \in \mathcal{H} \otimes \mathcal{G}$ such that $\mathrm{Tr}_{\mathcal{G}}[|\psi_\rho\rangle\langle\psi_\rho|] = \rho$. The Schmidt decomposition is useful for explicitly constructing such purifications:

f) Give a purification of an arbitrary quantum state ρ in terms of its eigenvalues and eigenvectors.

g) Show that two purifications $|\psi_1^\rho\rangle$ and $|\psi_2^\rho\rangle$ of the same state ρ are related by a unitary transformation that acts on $\mathcal{G}$ only.

2. **General teleportation schemes**

   In the lecture you saw a teleportation scheme using a maximally entangled state shared by Alice and Bob. In this exercise we will generalise this setting to teleportation schemes with higher local dimensions.

   We begin by reformulating the qubit teleportation scheme in terms of Bell-basis measurements. The Bell basis for two qubits is given by

   $$|\Phi_0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), |\Phi_1\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle),$$
   $$|\Phi_2\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), |\Phi_3\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

   a) Show that the Bell basis can be prepared starting from $|\Phi_0\rangle$ using local Pauli operations only.

   This reformulation generalises to a d-dimensional teleportation scheme in which Alice and Bob share a maximally entangled state $|\omega\rangle = \frac{1}{\sqrt{d}}\sum_{i=1}^{d}|ii\rangle$. As above the scheme is based on measuring in a maximally entangled orthonormal basis set $\{|\Psi_\alpha\rangle\}_{\alpha=1}^{d^2}$, i.e., an orthonormal basis for which $\mathrm{Tr}_1[|\Psi_\alpha\rangle\langle\Psi_\alpha|] = \mathbb{1}_d = \mathrm{Tr}_2[|\Psi_\alpha\rangle\langle\Psi_\alpha|]$.

   There exist several constructions of linearly independent sets $\{U^\alpha\}_{\alpha=1}^{d^2}$ of d^2 trace-wise orthogonal unitary operators $U^\alpha \in U(d)$,

   $$\mathrm{Tr}[U^{\alpha\dagger}U^\beta] = \mathrm{Tr}[U^{\beta\dagger}U^\alpha] = \delta_{\alpha\beta}\mathbb{1}$$

   for all α and β. In the following, we just assume the existence of such a set.

   b) Show that such a set $\{U^\alpha\}_{\alpha=1}^{d^2}$ gives rise to a maximally entangled basis set by setting

   $$|\Psi_\alpha\rangle = U^\alpha \otimes \mathbb{1}\,|\omega\rangle\,.$$

   c) Use the completeness relation for $\{|\Psi_\alpha\rangle\}_\alpha$ to show that any such operator basis satisfies

   $$\frac{1}{d}\sum_\alpha U^\alpha_{ij}\overline{U}^\alpha_{kl} = \delta_{ik}\delta jl. \tag{1}$$

   d) Expand the basis states $|\Psi_\alpha\rangle$ in the computational product basis $\{|ij\rangle\}_{ij}$.

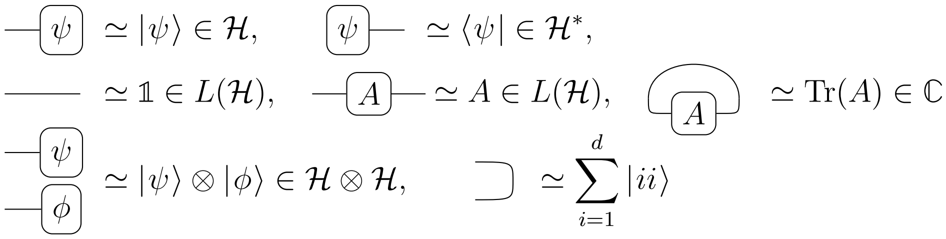
   Now consider the setting in which Alice and Bob share the state $|\omega\rangle$ and Alice measures her part of the system in the basis $|\Psi_\alpha\rangle$.

   e) Insert the resolution of the identity $\sum_\alpha |\Psi_\alpha\rangle\langle\Psi_\alpha|$ and use the result from (d) to derive the unitary corrections required in the d-dimensional teleportation scheme.

3. **Introduction to graphical calculus with tensor networks**

   As you might have noticed, already for a little number of tensor factors even simple calculations can become hard to follow quite easily. Hence, an alternative approach to visualize such calculations was developed. Namely, graphical calculus with tensor networks, often atributed to Roger Penrose. We will give a short introduction into the basics of this calculation technique in this exercise. However, we encourage you to have a look into `https://arxiv.org/pdf/1603.03039.pdf`, which gives a nice and complete overview over tensor networks. For this course, you won't need most of the content but it constitutes a good reference where you can find any concept we will use (in particular, in chapter 1 and 2).

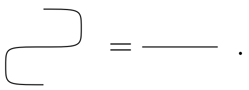
In tensor networks, we have the following correspondences



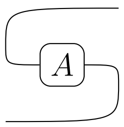
One can think of each unconnected leg carrying a (dual) Hilbert space.

a) Draw the expectation value $\langle\psi| A |\psi\rangle$ as a tensor network.

b) Proof



c) Proof that the network



corresponds to the transpose of opertor A.

d) Consider an operator B acting on two a bi-partite Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. Draw the tensor network implementing the partial transpose on one of the subsystems.

e) Proof the equality $\mathrm{Tr}(A^2) = \mathrm{Tr}((A \otimes A)F)$ using tensor networks. F denotes the *flip operator* exchanging the two subsystems, i.e. $F : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}, |i\rangle \otimes |j\rangle \mapsto |j\rangle \otimes |i\rangle$.

How much more annoying would this proof be without the graphical calculus?