Freie Universität Berlin

Tutorials on Quantum Information Theory

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Problem Sheet 5

Channel representations and Norms for matrices part II

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1. **Equivalence between representations of quantum channels**
   Let us first show that the Choi-Jamiołkowski map $J : L(L(\mathcal{X}), L(\mathcal{Y})) \to L(\mathcal{Y} \otimes \mathcal{X})$ is a linear bijection between the CPT maps on the one hand and the set of quantum states on $\mathcal{Y} \otimes \mathcal{X}$ with partial-trace over $\mathcal{Y}$ being maximally mixed on the other hand.

   a) Show that the inverse map can be defined by $\tilde{T}(X) = \mathrm{Tr}_{\mathcal{X}}[J(T)(\mathbb{1}_{\mathcal{Y}} \otimes X^T)]$ and makes J a bijection as described above.

   Let $\rho_T \in \mathcal{Y} \otimes \mathcal{X}$ be the Choi-Jamiołkowski state corresponding to the quantum channel T.

   b) Determine a set of Kraus operators representing T.

   c) Determine a unitary U_T representing T via the Stinespring representation.

   Now, let U_T be a unitary representing T in the Stinespring representation.

   d) Determine the Choi-Jamiołkowski state representing T.

   The rank of a quantum channel is defined as the rank of its Choi matrix.

   e) Show that a quantum channel with rank r can be represented as a Stinespring dilation using an auxiliary system of dimension r.

2. **Examples of quantum channels**
   Now we are ready to look at some examples of quantum channels acting on qubits, i.e., $\mathcal{H} = \mathbb{C}^2$. The following maps are important so-called noise channels

   $$F_\epsilon(A) := \epsilon XAX + (1-\epsilon)A$$
   $$D_\epsilon(A) := \epsilon\, \mathrm{Tr}[A]\frac{\mathbb{1}}{d} + (1-\epsilon)A$$
   $$A_\epsilon(A) := \epsilon\, \mathrm{Tr}[A]\, |0\rangle\langle 0| + (1-\epsilon)A,$$

   where $\epsilon \in [0,1]$.

   a) For each channel, show that it is CPT.

   b) For each channel, give its Choi-Jamiołkowski state, a Kraus representation and a Stinespring representation.

   Hint: It may help to consider $\epsilon = 1$ *in a first step and then generalize to arbitrary* $\epsilon \in [0,1]$.

   c) Give a physical interpretation and a good name for each channel.

3. **Schatten p-norms**
   On the last excercise sheet we have studied the ℓ_p-norms on vector spaces. The ℓ_p-norms have important cousins on matrix spaces, the Schatten p-norms. As they are important distant measures in quantum information, we study there different definitions and properties in this excerice.

   One way to introduce the Schatten p-norm with $p \in [1,\infty)$ for a matrix $A \in \mathbb{C}^{n\times n}$ is

   $$||A||_p := \left(\mathrm{Tr}\left[|A|^p\right]\right)^{\frac{1}{p}}, \tag{1}$$

where $|A| := \sqrt{A^\dagger A}$ is the matrix absolute value. Furthermore, the case $p = \infty$ is defined as the limit $||A||_\infty = \lim_{p\to\infty} ||A||_p$.

These norms are related to the ℓ_p-norms of the eigenvalues (or more generally the singular values) of A.

a) Let A be a Hermitian matrix and let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be the vector of its eigenvalues. Show that

$$||A||_p = ||\lambda||_{\ell_p} \tag{2}$$

for all p.

With this characterisation we have also established that the Schatten p-norms are invariant under unitary transformations.

b) Give the statement and proof for the Hölder inequality for Schatten p-norms.

Hint: Actually, proving the Hölder inequality rigorously involves proving the "von Neumann-inequality", which turns out to be quite intricate. In this exercise you can simply use it:
Let A and B be two matrices and let $s(A)$ and $s(B)$ be the vector of singular values of A and B, respectively, ordered decreasingly. Then it holds that

$$|\operatorname{Tr}[AB]| \le \operatorname{Tr}|AB| \le \sum_i s_i(A) s_i(B). \tag{3}$$

For a proof (sketch) of this inequality, you can take a look at Bhatia's book on matrix analysis. Slightly more direct proofs using doubly stochastic matrices were worked out by Mirsky. A more elementary proof was given R. D. Grigorieff in a note in '92. You can find it on his webpage.

The most important Schatten p-norms have other interesting expressions:

c) Show that the Schatten 2-norm or Frobenius norm fulfils

$$||A||_2^2 = \sum_{i,j=1}^{n} |A_{ij}|^2. \tag{4}$$

In general, one can define the operator norms induced by the ℓ_p-norms:

$$||A||_{\ell_p \to \ell_q} = \sup_{||x||_{\ell_p}=1} ||Ax||_{\ell_q}. \tag{5}$$

d) What is the Schatten p-norm equal to $||\cdot||_{\ell_2\to\ell_2}$?

Another important properties of Schatten p-norms is *sub-multiplicativity*, $||AB||_p \le ||A||_p ||B||_p$ for all p and $A, B \in \mathbb{C}^{n\times n}$. Sometimes the term *matrix norm* is exclusively used for sub-multiplicative norms on matrix spaces.

e) Show the sub-multiplicativity of the Schatten p-norms.