Freie Universität Berlin

Tutorials on Quantum Information Theory

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Problem Sheet 6

Entanglement witnesses

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1. **An entanglement witness for the maximally entangled state**
   Let $|\Omega\rangle$ be the maximally entangled state.
   a) Show that $W = \mathbb{1} - d\,|\Omega\rangle\langle\Omega|$ is a witness for $|\Omega\rangle$.
   b) Give an example of an entangled state that is not detected by W.

2. **The reduction map as a witness**
   The *reduction map* is defined as $\Lambda_R(X) = \mathrm{Tr}(X)\mathbb{1} - X$.
   a) Show that Λ_R is positive but not completely positive, in other words it is a witness.
   b) Give at least one example for states that are detectable by Λ_R.
   c) Give at least one example for entangled states that are not detected by Λ_R.
   d) What is the observable witness associated to the positive map by the Choi-Jamiolkowski-isomorphism?

   A map Λ is called *decomposable* if it can be written as $\Lambda = P_1 + P_2 \circ T$, where P_1, P_2 are completely positive maps and T is the transpose.
   e) Show that any state that is detected by Λ_R can also be detected by the partial transpose criterion.
   Hint: Argue first that Λ_R is decomposable.
   f) Translate the condition of a map Λ being decomposable to a criterion for the observable witness $J(\Lambda^*)$. What is the implication of a self-adjoint observable witness being decomposable in this sense?

3. **Constructing entanglement witnesses from the partial transpose**
   In the entanglement theory of bi-partite systems the partial transpose criterion plays a prominet role. Let $T : L(\mathcal{H}) \to L(\mathcal{H})$ be the transposition map $X \mapsto T(X) = X^T$. The partial transpose is the map $T : L(\mathcal{H} \otimes \mathcal{H}) \to L(\mathcal{H} \otimes \mathcal{H})$. Let $(\cdot, \cdot)$ be the Hilbert-Schmidt inner-product on $L(\mathcal{H})$ defined as $(X, Y) = \mathrm{Tr}(X^T Y)$. The adjoint Λ^* of a map $\Lambda : L(\mathcal{H}) \to L(\mathcal{H})$ with respect to $(\cdot, \cdot)$ is defined such that $(\Lambda(X), Y) = (X, \Lambda^*(Y))$ holds for all $X, Y \in L(\mathcal{H})$.

   In the lecture, we saw that for any positive but not completely positive map Λ, $J(\Lambda^*)$ is also witness. But $J(\Lambda^*)$ is not necessarily detecting all the states that Λ detects. Let Λ detect ρ_e and let $|\eta\rangle$ be an eigenvector with negative eigenvalue of $(\mathbb{1} \otimes \Lambda)(\rho_e)$. $|\Omega\rangle$ shall be the maximally entangled state.
   a) Show that $\mathbb{1} \otimes T$ is self-adjoint, i.e. $(\mathbb{1} \otimes T)^* = \mathbb{1} \otimes T$.
   b) Show that $J(\Lambda^*)$ detects $\hat{\rho}_e = (\mathbb{1} \otimes X^\dagger)\rho_e(\mathbb{1} \otimes X)$ where X is defined such that $|\eta\rangle = \mathbb{1} \otimes X\,|\Omega\rangle$.

   With this we have established the missing direction in the proof of the lectures' theorem relating positive maps and observables as witnesses.

   It is also possible to construct an observable that detects ρ_e itself from Λ. To this end, we define $\mathcal{W}_e = (\mathbb{1} \otimes \Lambda^*)(|\eta\rangle\langle\eta|)$.

c) Show that this construction, in fact, gives rise to an entanglement witness $\mathcal{W}_e$ for ρ_e.

As an application of this construction we consider the following setting. In our (fictitious) lab, we are trying to prepare a two-qubit state $|\psi\rangle \in \mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$. We use a simple model[1] for what is actually happening in the lab, namely that we prepare a state with some noise

$$\rho(p) := p\,|\psi\rangle\langle\psi| + (1-p)\frac{\mathbb{1}}{4}.$$

Our goal is to have an observable witness that decides whether $\rho(p)$ is entangled or not. To this end, we will use the fact that for two-qubits system there exist no entangled PPT states. Therefore, if $\rho(p)$ is entangled, the partial transpose $\mathbb{1} \otimes T$ will always detect $\rho(p)$..

d) Assume $|\psi\rangle$ has Schmidt decomposition $|\psi\rangle = a\,|01\rangle + b\,|10\rangle$. Determine the values of p depending on a, b such that $\rho(p)$ is entangled.

e) Use the eigenvector corresponding to a negative eigenvalue of $(\mathbb{1} \otimes T)(\rho(p))$ in order to derive an entanglement witness $\mathcal{W}$ for $\rho(p)$.

f) Show that, in fact, the witness $\mathcal{W}$ detects *all* entangled states of the form $\rho(p)$.

Happy Holidays everyone. Take some time off! ☺

[1] What is the corresponding noise channel for this model?