Freie Universität Berlin
Tutorials on Quantum Information Theory
Winter term 2020/21

Problem Sheet 11
Measurement based quantum computing

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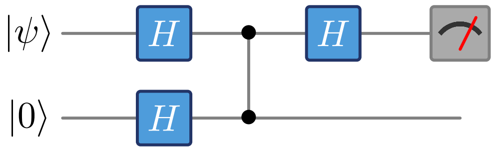
1. **Gate teleportation**
At the heart of measurement based quantum computing is the principle of gate teleportation. A simple version of this can be demonstrated in the following circuit:

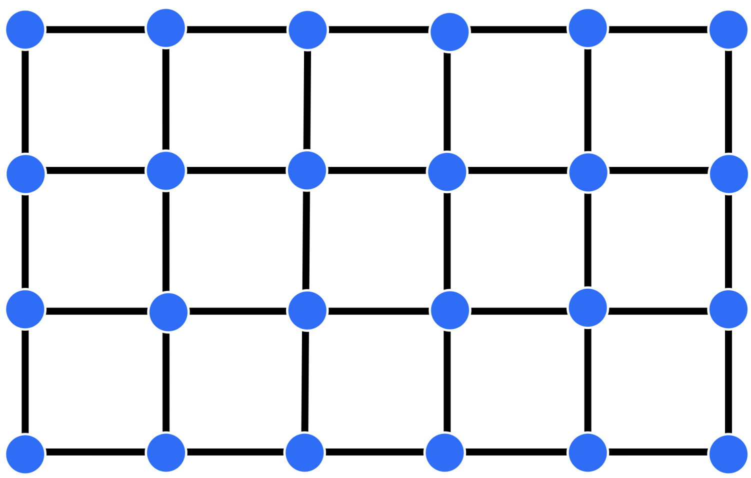


   a) In the above circuit we measure one of the registers in the Z eigenbasis. What is the resulting state on the remaining subsystem depending on the measurement outcome?

   b) Can you generalize this set-up to a "wire"? That means a measurement based processed that transports quantum information on a 1D line?

   c) Consider a graph state on a 1D line and measure each qubit either in the eigenbasis of $e^{\mathrm{i}\phi_i Z} X e^{-\mathrm{i}\phi_i Z}$, where $\phi_i \in [0, 2\pi)$ is drawn uniformly randomly independently for each qubit. What is the effective circuit acting on the last qubit?

2. **Universal Quantum Computation with Cluster States** In this exercise we consider the two dimensional cluster state. It lies at the core of measurement based quantum computation (MBQC) because, once prepared, one can perform *any* quantum computation on it with single qubit measurements (in various bases!). In this exercise we will show how an arbitrary quantum operation is performed on such a state by mapping it to the circuit model, which might be the more intuitive way for you to understand a quantum computation. But first, let us fix some definitions.



The 2D cluster state is defined by a graph state on a 2D square grid $\mathcal{L}$ (see figure above). This means, we prepare the $L_x \times L_y$ qubits in $|+\rangle$. At this point, the qubits are in a simple product state $\bigotimes_{i\in\mathcal{L}} \frac{1}{2}(|0\rangle_i + |1\rangle_i)$. To create the entanglement needed for any useful quantum computation, $CZ_{i,j}$ (controlled Z gate) is applied to every pair of adjacent qubits (i, j). Any cluster state can be prepared in this way and as we will for the main part of this exercise work with graph states on different graphs and only in the end see how all these can be obtained from the graph state defined on a square grid.

A probably simpler way to understand a cluster/graph state is by its stabilizers. The cluster state $|\mathcal{L}\rangle$ is stabilized by the set of stabilizers (and by all its combinations, i.e. the group generated by them)

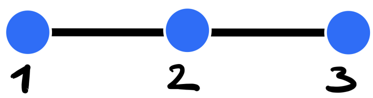
$$\mathcal{S} = \{X_a \prod_{i \sim a} Z_i \mid a \in \mathcal{L}\},$$

where $i \sim a$ denotes the set of qubits adjacent to qubit a. In paticular, it holds that $S_a |\mathcal{L}\rangle = |\mathcal{L}\rangle \ \forall S_a \in \mathcal{S}$.

a) Proof that the state prepared with the above procedure is indeed stabilized by $\mathcal{S}$. *Hint: You can use the stabilizer formalism introduced on sheet 9.*

In exercise 1 we already saw how an arbitrary 1-qubit gate in $SU(2)$ can be performed with a one-dimensional graph state.

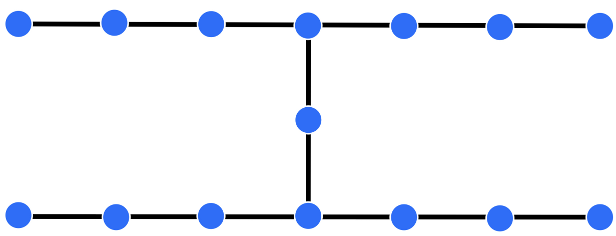
b) As a recap, construct a protocol to implement a $\pi/2$ Z-rotation with the 3-qubit graph state,



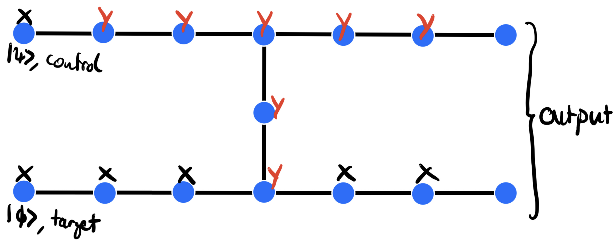
i.e. after all the measurement(s), qubit 3 should be in the state $e^{i\frac{\pi}{2}Z}|i\rangle$, where $|i\rangle$ is the state into which qubit 1 is prepared in the beginning of the protocol.

In this exercise we will proof how one can perform an entangling gate on a two-dimensional graph state und with that achieve universality with MBQC on a 2D cluster state.

c) Consider the following graph state:



Show that the following measurements implement a CNOT gate between the two input states $|\psi\rangle$ and $|\phi\rangle$ up to local pauli corrections:



Depending on the measurement outcomes, one has to perform Pauli corrections to obtain the pure CNOT. Which are they?
Hint: There are two ways to proof this. Either, one explicitly calculates the output of the full circuit corresponding to the preparation and the measurements or one uses the stabilizer formalism where one only has to keep track how the stabilizers of the graph state change during the measurements.

d) Show that all the individual operations introduced until now (one-qubit rotations and CNOT) compose according to their compostion in the circuit model. What operation has to be performed on the qubits where the graphs are "stitched together"?

Until now we have only shown that given a graph state on a graph that already has the connectivity of the corresponding circuit in the circuit model can simulate the given circuit. To complete our proof, it remains to be shown that any such state can be obtained from a 2D cluster state, the graph state on a 2D grid.

e) Show that, given a 2D cluster state, one can "cut out" any 2D network and obtain the corresponding graph state by single qubit Z measurements on the qubits that should not participate in the computation.