Freie Universität Berlin
Tutorials on Quantum Information Theory
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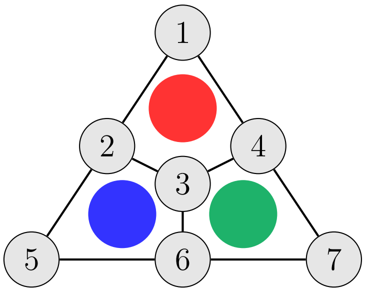
Problem Sheet 12
Basic Quantum Error Correcting Codes

J. Eisert, J. Haferkamp, J. C. Magdalena De La Fuente

Review the old exercise sheets and good luck with the exam!

1. **7-qubit (Steane) Code**
A convenient class of quantum codes are Calderbank–Shor–Steane (CSS) codes. In CSS codes, the generators of the stabilizer group (parity check operators) of the code are pure products of Z or X operators, so they do not contain any mixed terms. which would be proportional to Y, or even Non-Pauli terms.[1] In this exercise we will introduce and investigate simple properties of the simplest CSS code, the 7-qubit code.



The 7-qubit codes uses 7 physical qubits to encode logical qubit(s). It is graphically represented in the figure above. The nodes correspond to the qubits. The structure of the graph will become clear once we have defined the stabilizer generators $\{S_1, S_2, ..., S_6\}$.

qubit	1	2	3	4	5	6	7
S_1	Z	Z	Z	Z	$\mathbb{1}$	$\mathbb{1}$	$\mathbb{1}$
S_2	$\mathbb{1}$	Z	Z	$\mathbb{1}$	Z	Z	$\mathbb{1}$
S_3	$\mathbb{1}$	$\mathbb{1}$	Z	Z	$\mathbb{1}$	Z	Z
S_4	X	X	X	X	$\mathbb{1}$	$\mathbb{1}$	$\mathbb{1}$
S_5	$\mathbb{1}$	X	X	$\mathbb{1}$	X	X	$\mathbb{1}$
S_6	$\mathbb{1}$	$\mathbb{1}$	X	X	$\mathbb{1}$	X	X

The operators in the table are tensor products of single qubite X or Z operators and jointly act on the qubits adjacent to each face in the above graph.

a) Show that the stabilizer group $\mathcal{S} = \langle S_1, ..., S_6 \rangle$ is Abelian. From this follows that the generators can simultaneously diagonalized. What is the dimensionality k of each of the eigenspaces?

b) The logical space, the *codespace* C, is defined by the common $+1$ eigenspace of all S_i. Write down a basis for that space $\mathcal{B}_C = \{|\bar{a}\rangle \mid a = 0, ..., k-1\}$ (which can be defined as the logical computational basis). Find corresponding logical operators

$$\bar{Z} : |\bar{a}\rangle \mapsto (-1)^a |\bar{a}\rangle \tag{1}$$
$$\bar{X} : |\bar{a}\rangle \mapsto |\overline{a \oplus 1}\rangle , \tag{2}$$

where $\oplus$ denotes binary addition.

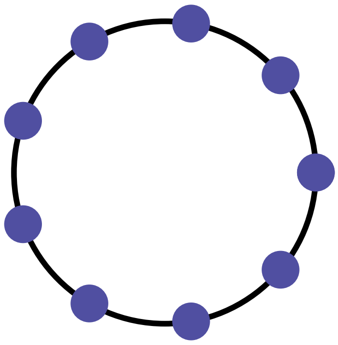
[1] In fact, the X part, as well as the Z part, alone can be viewed as a classical linear code.

c) What is the distance of the code, i.e. the smallest weight non-trivial logical operator?
Hint: The logical operators you found before are not unique!

d) *(extra exercise)* Find an operator implementing the logical Hadamard transformation. What is special about this operator?
It suffices to consider the stabilizer generators, not the logical basis states.

2. **Repetition Code and Toric Code**
The repetition code is the simplest classical code one can think of. It uses n bits to encode 1 logical bit. In the language of stabilizers, it is defined by the generators $S_i = Z_i Z_{i+1}$. For simplicity, let us assume periodic boundary conditions, i.e. $S_n = Z_n Z_1$. In the first part of this exercise, we will understand the classical bitstrings as states in a vector space (just as for quantum codes). For now, you can think of a classical bit as a qubit we only have access to one basis, e.g. the Z basis.

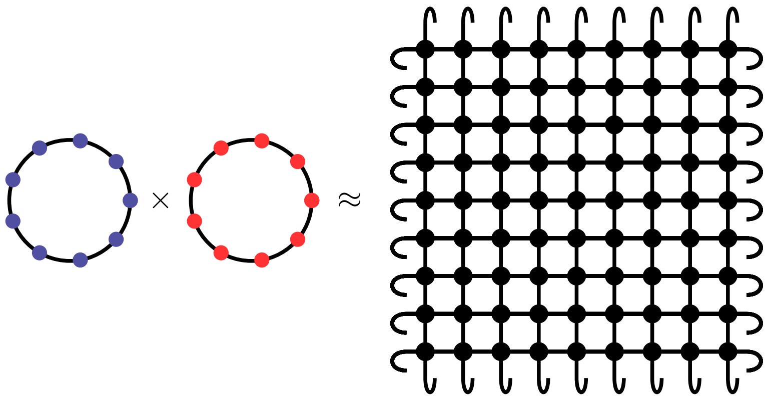
We can depict the repetition code (with periodic boundary conditions) as a discretized circle (one-dimensional sphere S^1),



In this representation, each node corresponds to a Z-stabilizer and every edge to a qubit.[2]

a) What are the classical codewords, corresponding to basis states of the code? Which type of errors can be detected with this code? What distance would you assign to it (as a quantum/classical code)?

As you might have noticed already in the stabilizers, we are missing "half" of them, namely the X stabilizers, to have a proper quantum code. However, we can still use the above classical code to define a quantum code. This is done by taking a so called product of two repetition codes, one defined with Z stabilizers and one with Xs at the same places. Pictorialy, this product reads



On the left hand side, we denoted the Z stabilizers in blue and the X stabilizers in red. On the right hand side, the periodic boundary conditions carry over to two dimensions

[2]You could also see it the other way around but our convention is better suited for the construction in this exercise.

and we obtain a cellulation of the two-torus $T^2 = S^1 \times S^1$ which can be seen as a product space itself. We have yet to figure our how to consistently define the stabilizers on the torus. It turns out that the picture above is a bit too simplified to see it just from the right graph. The detailed construction goes beyond of the scope of this exercise but we hope that you could at least gain an intuition for this product.

The quantum code on one obtains on the right is called the *toric code* and is one of the most studied quantum codes today. In the toric code, the qubits are placed on the edges of a cellulation of a torus (see right). For each vertex v and for each face f of the cellulation, one defines a stabilizer generator

$$S_v = \prod_{i \sim v} X_v \quad \text{and} \quad S_p = \prod_{i \in \partial p} Z_i, \tag{3}$$

called *vertex terms* and *plaquette terms.* $\sim$ denotes "adjacent to" and ∂ the "boundary of".

b) Show that $\mathcal{S} = \langle \{S_v, S_p\} \rangle$ is Abelian.

c) How many qubits does the toric code encode? What are the logical operators $\overline{X}$ and $\overline{Z}$? How do I rotate the logical basis, i.e. what operator implements the Hadamard gate on the codespace?

d) What is the distance of the code?

e) How does a single qubit Pauli error X, respectively Z, affect the eigenstates of the stabilizers? How can it be detected?