Freie Universität Berlin

Tutorials on Quantum Information Theory

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Problem Sheet 0

Warm-up

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1. **Tensor products**

The configuration space of a quantum system with multiple degrees of freedom is described by the tensor product of the Hilbert spaces of each degree of freedom. In the following exercise we will familiarise ourselves with the construction of tensor product spaces.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces with basis $B_1 = \{|i\rangle_1\}_{i=1}^d$ and $B_2 = \{|j\rangle_2\}_{i=1}^D$, respectively. One can construct a new vector space $\mathcal{H}_1 \otimes \mathcal{H}_2$ by using the set of tuples $B_1 \times B_2 = \{(|i\rangle_1, |j\rangle_2) : |i\rangle_1 \in B_1, |j\rangle_2 \in B_2\}$ as a basis. The basis elements $(|i\rangle_1, |j\rangle_2)$ are also typically denoted by $|i\rangle |j\rangle$, $|i,j\rangle$ or $|i\rangle \otimes |j\rangle$. The last notation can be extended to a bilinear composition $\otimes : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_1 \otimes \mathcal{H}_2$ by defining

$$|\psi\rangle \otimes |\phi\rangle \coloneqq \sum_{i=1}^{d} \sum_{j=1}^{D} \langle i|\psi\rangle \langle j|\phi\rangle |i,j\rangle . \tag{1}$$

a) What is the dimension of the vector space $\mathcal{H}_1 \otimes \mathcal{H}_2$? What is the Hilbert space of a system of n spin-1/2 particles? What is its dimension?

Solution: The dimension of $\mathcal{H}_1 \otimes \mathcal{H}_2$ is the number of pairs $|i,j\rangle$, hence dD. A spin 1/2-particle is described by the Hilbert space $\mathbb{C}^2$. Therefore, the space of n spin 1/2 particles is the n-fold tensor product $(\mathbb{C}^2)^{\otimes n} := \mathbb{C}^2 \otimes ... \otimes \mathbb{C}^2$ with dimension 2^n.

b) Show that the operation $\otimes : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_1 \otimes \mathcal{H}_2$ defined above is bilinear.

Solution: This follows directly from the linearity of $\langle i|\psi\rangle$. We show additivity in the first entry to demonstrate this:

$$(|\psi_1\rangle + |\psi_2\rangle) \otimes |\phi\rangle = \sum_{i=1}^{d} \sum_{j=1}^{D} \langle i|(|\psi_1\rangle + |\psi_2\rangle) \langle j|\phi\rangle |i,j\rangle \tag{2}$$

$$= \sum_{i=1}^{d} \sum_{j=1}^{D} (\langle i|\psi_1\rangle \langle j|\phi\rangle + \langle i|\psi_1\rangle \langle j|\phi\rangle) |i,j\rangle \tag{3}$$

$$= \sum_{i=1}^{d} \sum_{j=1}^{D} \langle i|\psi_1\rangle \langle j|\phi |i,j\rangle\rangle + \sum_{i=1}^{d} \sum_{j=1}^{D} \langle i|\psi_1\rangle \langle j|\phi\rangle) |i,j\rangle \tag{4}$$

$$= |\psi_1\rangle \otimes |\phi\rangle + |\psi_2\rangle \otimes |\phi\rangle . \tag{5}$$

c) Is $\otimes : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_1 \otimes \mathcal{H}_2$ surjective? (Please argue.)

Solution: No, this is not the case. It is easy to find counter-examples (such as the Bell state). The states that are not in the image of this map are precisely the pure entangled state.

d) Show that the following identities hold for all operators $A, B, C : \mathcal{H}_1 \to \mathcal{H}_1$ and vectors $|\phi\rangle, |\psi\rangle \in \mathcal{H}_1$:

(i) $(A \otimes B)(|\phi\rangle \otimes |\psi\rangle) = (A|\phi\rangle) \otimes (B|\psi\rangle)$

(ii) $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$

Solution: We have a straightforward definition of the tensor product of two operators similar to (1):

$$A \otimes B := \sum_{i,j} \sum_{l,k} \langle i|A|j\rangle \langle l|B|k\rangle |i,l\rangle\langle j,k|. \tag{6}$$

We can verify the first condition as follows. We start with the right hand side:

$$\begin{aligned}
(A|\phi\rangle) \otimes (B|\psi\rangle) &= \sum_{i,j} \langle i|A\psi\rangle \langle j|B\phi\rangle |i,j\rangle & (7)\\
&= \sum_{i,j}\sum_{l,k}\sum_{r,t} \langle i|l\rangle \langle l|Ak\rangle \langle k|\phi\rangle \langle j|m\rangle \langle t|\psi\rangle \langle r|Bt\rangle |i,j\rangle & (8)\\
&= \sum_{l,k}\sum_{r,t} \langle l|Ak\rangle \langle k|\phi\rangle \langle t|\psi\rangle \langle r|Bt\rangle |l,r\rangle & (9)\\
&= \sum_{l,k}\sum_{r,t} \langle l|Ak\rangle \langle k|\phi\rangle \langle t|\psi\rangle \langle r|Bt\rangle |l,r\rangle \langle k,r|(|\psi\rangle \otimes |\phi\rangle) & (10)\\
&= (A \otimes B)(|\phi\rangle \otimes |\psi\rangle). & (11)
\end{aligned}$$

The second condition follows from a similar calculation.

2. **Pauli matrices and the Bloch sphere**

The Pauli matrices are one of the most ubiquitous objects in quantum mechanics. They act on the simplest non-trivial Hilbert space $\mathcal{H} = \mathbb{C}^2$.

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In this exercise we want to recap their properties.

a) Show that these matrices mutually anticommute, i.e. $AB = -BA$ and that all of them square to the identity.

b) Explicitly compute the 4×4 matrices $X \otimes X$, $Z \otimes Z$, $X \otimes Y$ and $Y \otimes X$ in the tensor product basis.

c) Can you express the product XZ again as a Pauli matrix?

We now want to use Pauli matrices to study the space of all qubit observables: the hermitian matrices $h(\mathbb{C}^2)$. This space is canonically equipped with the *Hilbert-Schmidt inner product*

$$\langle A, B\rangle := \mathrm{Tr}(AB^\dagger).$$

The norm that is defined by this product is also called *Frobenius norm*. Both will constant companions in this course.

d) Show that with respect to this inner product that the Pauli matrices together with the identity form an orthogonal basis for $h(\mathbb{C}^2)$.

e) Find the normalized version of this basis with respect to the Frobenius norm.

Recall that a density matrix ρ is a hermitian operator with all positive eigenvalues such that $\mathrm{Tr}(\rho) = 1$. We restrict ourselves – for now – to the qubit case.

f) You have shown that the Paulis with the identity form a basis of the hermitian matrices. Prove that, in this basis, the set of density matrices is described by a unit ball $B_1(0) = \{(a, b, c) \in \mathbb{R}^3; a^2 + b^2 + c^2 \leq 1\}$ (called *Bloch sphere*).

g) Where do the pure states ($\rho = |\psi\rangle\langle\psi|$) live in this ball? Which point corresponds to the maximally mixed state ($\rho = \mathbb{1}/2$)?

3. **Beam splitters and interferometers**

In this exercise we consider a simple model of an interferometer. A key ingredient for most interferometers is a *beam splitter*, an optical device with two inputs and two outputs. An arbitrary input state as well as the corresponding output state can be modelled as a quantum state in $\mathcal{H} = \mathbb{C}^2$. A proper quantum beam splitter is modelled by a *linear, unitary operator* on the above Hilbert space such that the transmitted, respectively reflected, output state is measured with a certain probability p. Moreover, we assume that it acts the same on both input basis states.

a) Confirm that a 50:50 beam splitter (reflecting and transmitting each basis state with the same probability of 1/2) is described by the matrix

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}.$$

In the following we will consider such a 50:50 beam splitter if not stated otherwise. One only needs two beam splitters to build a simple interferometer. For example, consider a source that outputs a pure quantum state which we denote by $|0\rangle$. This state is then inputted into a beam splitter, creating a superposition of the two orthogonal output states. These two output states are then fed into a second beam splitter as inputs.

b) Calculate the probabilities that each of the output states of the second beam splitter is detected.

Now assume that we repeat this construction, namely we keep adding beam splitters that each take the output state of the preceding one as their inputs.

c) Calculate the detection probabilities of the output states of such an interferometer with N beam splitters, where $N \in \mathbb{N}$.

d) Let us go back to the orignial interferometer consisting of two beam splitters. What changes if we block one of the two intermediate paths, i.e. only one of the outputs of the first beam splitter reaches the second one?

Let us now consider more general beam splitters.

e) Construct a matrix $\tilde{S}(p)$ that models an *antisymmetric beam splitter* in the sense that each input basis state gets transmitted with a probability p instead of 1/2. Would it be possible to have a beam splitter whose transmission probability depends on the input state? If yes, how would the corresponding matrix look like? *Hint: Recall that any closed system quantum dynamics is unitary.*