Freie Universität Berlin
Tutorials on Quantum Information Theory
Winter term 2020/21

Problem Sheet 3
Teleportation and Introduction to Graphical Calculus

J. Eisert, J. Haferkamp, J. C. Magdalena De La Fuente

1. **Schmidt decomposition and purification**

In the lecture, you already saw the Schmidt decomposition of bipartite quantum states $|\Psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$ as given by

$$|\Psi\rangle = \sum_{i=1}^{d} \sqrt{\lambda_j} \, |\psi_j^1\rangle \, |\psi_j^2\rangle ,$$

where $\{|\psi_j^i\rangle\}$ are orthonormal bases of $\mathcal{H}_i$.

In this exercise, we will study some useful properties and applications of the Schmidt decomposition. To begin with, let us look at states with the same Schmidt coefficients, that is

$$|\Psi\rangle = \sum_{i=1}^{d} \sqrt{\lambda_j} \, |\psi_j^1\rangle \, |\psi_j^2\rangle , \quad |\Phi\rangle = \sum_{i=1}^{d} \sqrt{\lambda_j} \, |\phi_j^1\rangle \, |\phi_j^2\rangle .$$

a) Show that $|\Psi\rangle$ and $|\Phi\rangle$ are related by a local unitary, i.e., a unitary of the form $U \otimes V$ with U and V unitary. Give that unitary explicitly.

Solution:

$$|\Psi\rangle = \left(\sum_j |\psi_j^1\rangle \langle \phi_j^1| \right) \otimes \left(\sum_j |\psi_j^2\rangle \langle \phi_j^2| \right) |\Phi\rangle$$

b) Show that any local unitary transformation leaves the Schmidt coefficients invariant.

Solution:

$$U \otimes V \, |\Psi\rangle = \sum_{i=1}^{d} \sqrt{\lambda_j} \, U \, |\psi_j^1\rangle \otimes V \, |\psi_j^2\rangle ,$$

but since U is a fixed unitary $U\,|\psi_j^i\rangle$ is still an orthogonal basis, hence we have a new state with the same Schmidt coefficients.

This gives rise to a nice interpretation of the Schmidt coefficients of a state in terms of entanglement (more soon!):

c) Determine the reduced density matrices $\rho_1 = \mathrm{Tr}_2 \, |\Psi\rangle\langle\Psi|$ and $\rho_2 = \mathrm{Tr}_1 \, |\Psi\rangle\langle\Psi|$. How can the Schmidt coefficients be interpreted? What are the Schmidt coefficients of the maximally entangled state?

Solution:

$$\rho_1 = \mathrm{Tr}_2 \left[\sum_{ij} \sqrt{\lambda_i \lambda_j} \, |\psi_i^1\rangle\langle\psi_j^1| \, |\psi_i^2\rangle\langle\psi_j^2| \right]$$
$$= \sum_i \lambda_i \, |\psi_i^1\rangle\langle\psi_i^1|$$

The Schmidt coefficients are the eigenvalues of the reduced density matrix. the maximally entangled state has Schmidt coefficients $1/d$.

d) Use the Schmidt decomposition to show that *any* bipartite state $|\Psi\rangle$ can be expressed as

$$|\Psi\rangle = (X \otimes \mathbb{1})\,|\Omega\rangle\,,$$

where $|\Omega\rangle$ is a maximally entangled state.

Solution: Let $|\Psi\rangle$ have Schmidt decomposition as above. We then choose $|\Omega\rangle = \frac{1}{\sqrt{d}} \sum_i |\psi_i^2\rangle\,|\psi_i^2\rangle$, and $X = \sum_i \sqrt{d\lambda_i}\,|\psi_i^1\rangle\,\langle\psi_i^2|$.

The maximally entangled state is *invariant* under certain product unitaries $U \otimes V$.

e) What are the conditions on U and V for this to be the case?

Solution:

$$|\omega\rangle = U \otimes V\,|\omega\rangle \Leftrightarrow \frac{1}{\sqrt{d}} \sum_i U\,|i\rangle\,V\,|i\rangle = \frac{1}{\sqrt{d}} \sum_{ijk} U_{ji}\,|j\rangle\,V_{ki}\,|k\rangle = \frac{1}{\sqrt{d}} \sum_i |i\rangle\,|i\rangle$$

and hence $\sum_i U_{ji} V_{ki} = \sum_i U_{ji} V_{ik}^T = (UV^T)_{jk} = (VU^T)_{kj} = \delta_{jk}$. But this is the case *iff* $V^T = U^\dagger$ and hence $V = \overline{U}$.

Recall from the lecture that for any quantum state $\rho \in \mathcal{L}(\mathcal{H})$ there exists a pure quantum state $|\psi_\rho\rangle \in \mathcal{H} \otimes \mathcal{G}$ such that $\mathrm{Tr}_{\mathcal{G}}[|\psi_\rho\rangle\!\langle\psi_\rho|] = \rho$. The Schmidt decomposition is useful for explicitly constructing such purifications:

f) Give a purification of an arbitrary quantum state ρ in terms of its eigenvalues and eigenvectors.

Solution:

$$\rho = \sum_i \lambda_i\,|\psi_i\rangle\!\langle\psi_i| \Rightarrow |\psi_\rho\rangle = \sum_i \sqrt{\lambda_i}\,|\psi_i\rangle\,|\psi_i\rangle$$

g) Show that two purifications $|\psi_1^\rho\rangle$ and $|\psi_2^\rho\rangle$ of the same state ρ are related by a unitary transformation that acts on $\mathcal{G}$ only.

Solution: Let $|\psi_1^\rho\rangle$ and $|\psi_2^\rho\rangle$ be two purifications of the same state ρ, i.e. $\mathrm{Tr}_{\mathcal{G}}\,|\psi_1^\rho\rangle\!\langle\psi_1^\rho| = \rho = \mathrm{Tr}_{\mathcal{G}}\,|\psi_2^\rho\rangle\!\langle\psi_2^\rho|$.

We can write the Schmidt decomposition of $|\psi_1^\rho\rangle$ and $|\psi_2^\rho\rangle$ as

$$|\psi_1^\rho\rangle = \sum_i \sqrt{\lambda_i}\,|\psi_i\rangle\,|\psi_j^1\rangle$$
$$|\psi_2^\rho\rangle = \sum_i \sqrt{\lambda_i}\,|\psi_i\rangle\,|\psi_j^2\rangle\,,$$

which must hold for $\rho = \sum_i \lambda_i\,|\psi_i\rangle\!\langle\psi_i|$ since the eigendecomposition is unique and as we saw above, the Schmidt basis on the first component fully determines the reduced state.

But the two orthonormal bases $\{|\psi_j^1\rangle\}_j$, $\{|\psi_j^2\rangle\}_j$ on $\mathcal{G}$ are related via a unitary transformation that acts on $\mathcal{G}$ only.

2. **General teleportation schemes**

In the lecture you saw a teleportation scheme using a maximally entangled state shared by Alice and Bob. In this exercise we will generalise this setting to teleportation schemes with higher local dimensions.

We begin by reformulating the qubit teleportation scheme in terms of Bell-basis measurements. The Bell basis for two qubits is given by

$$|\Phi_0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), |\Phi_1\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle),$$
$$|\Phi_2\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), |\Phi_3\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

a) Show that the Bell basis can be prepared starting from $|\Phi_0\rangle$ using local Pauli operations only.

Solution:

$$\mathbb{1} \otimes Z |\Phi_0\rangle = |\Phi_1\rangle$$
$$\mathbb{1} \otimes X |\Phi_0\rangle = |\Phi_2\rangle$$
$$\mathbb{1} \otimes XZ |\Phi_0\rangle = |\Phi_3\rangle$$

Solution: In the lecture, you saw the scheme in which Alice applies $(H \otimes \mathbb{1}^{\otimes 2})(CX \otimes \mathbb{1})$ to $|\psi\rangle |\Phi_0\rangle$ and then measures in the Z-basis. She then communicates her results, say a, b on the two registers to Bob, who applies $X^a Z^b$ as a correction to obtain $|\psi\rangle$ on his side.

The two schemes are equivalent via the identification of outcomes

$$00 \leftrightarrow 0, 10 \leftrightarrow 1, 01 \leftrightarrow 2, 11 \leftrightarrow 3,$$

where we used (a).

This reformulation generalises to a d-dimensional teleportation scheme in which Alice and Bob share a maximally entangled state $|\omega\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |ii\rangle$. As above the scheme is based on measuring in a maximally entangled orthonormal basis set $\{|\Psi_\alpha\rangle\}_{\alpha=1}^{d^2}$, i.e., an orthonormal basis for which $\mathrm{Tr}_1[|\Psi_\alpha\rangle\langle\Psi_\alpha|] = \mathbb{1}_d = \mathrm{Tr}_2[|\Psi_\alpha\rangle\langle\Psi_\alpha|]$.

There exist several constructions of linearly independent sets $\{U^\alpha\}_{\alpha=1}^{d^2}$ of d^2 trace-wise orthogonal unitary operators $U^\alpha \in U(d)$,

$$\mathrm{Tr}[{U^\alpha}^\dagger U^\beta] = \mathrm{Tr}[{U^\beta}^\dagger U^\alpha] = \delta_{\alpha\beta}\mathbb{1}$$

for all α and β. In the following, we just assume the existence of such a set.

b) Show that such a set $\{U^\alpha\}_{\alpha=1}^{d^2}$ gives rise to a maximally entangled basis set by setting

$$|\Psi_\alpha\rangle = U^\alpha \otimes \mathbb{1} |\omega\rangle .$$

Solution: Maximally entangled is clear.

This is a basis using

$$\langle\Psi_\alpha|\Psi_\beta\rangle = \langle\omega| {U^\alpha}^\dagger U^\beta \otimes \mathbb{1} |\omega\rangle = \delta_{\alpha\beta}.$$

c) Use the completeness relation for $\{|\Psi_\alpha\rangle\}_\alpha$ to show that any such operator basis satisfies

$$\frac{1}{d} \sum_\alpha U^\alpha_{ij} \overline{U}^\alpha_{kl} = \delta_{ik}\delta_{jl}. \tag{1}$$

Solution: First, note that $U^\alpha \otimes \mathbb{1} |\omega\rangle = \sum_{ijk} \frac{1}{\sqrt{d}} U^\alpha_{ij} |i\rangle\langle j| |kk\rangle = \sum_{ik} \frac{1}{\sqrt{d}} U^\alpha_{ik} |ik\rangle$. We then demand the completeness relation

$$\mathbb{1} = \sum_\alpha |\Psi_\alpha\rangle\langle\Psi_\alpha| = \sum_\alpha U^\alpha \otimes \mathbb{1} |\omega\rangle\langle\omega| U^{\alpha\dagger} \otimes \mathbb{1}$$
$$= \frac{1}{d} \sum_{ijkl} U^\alpha_{ij} \overline{U}^\alpha_{kl} |ij\rangle \langle kl| = \sum_{ij} |ij\rangle\langle ij| ,$$

from which we conclude the claim.

d) Expand the basis states $|\Psi_\alpha\rangle$ in the computational product basis $\{|ij\rangle\}_{ij}$.

Solution: We have already seen $|\Psi_\alpha\rangle = U^\alpha \otimes \mathbb{1} |\omega\rangle = \sum_{ijk} \frac{1}{\sqrt{d}} U^\alpha_{ij} |i\rangle\langle j| |kk\rangle = \sum_{ik} \frac{1}{\sqrt{d}} U^\alpha_{ik} |ik\rangle$.

Now consider the setting in which Alice and Bob share the state $|\omega\rangle$ and Alice measures her part of the system in the basis $|\Psi_\alpha\rangle$.

e) Insert the resolution of the identity $\sum_\alpha |\Psi_\alpha\rangle\langle\Psi_\alpha|$ and use the result from (d) to derive the unitary corrections required in the d-dimensional teleportation scheme.

Solution: Using the expansion of $|\Psi_\alpha\rangle$ and $|\psi\rangle$ in the computational basis, we obtain

$$|\psi\rangle |\omega\rangle = \sum_\alpha |\Psi_\alpha\rangle\langle\Psi_\alpha| |\psi\rangle |\omega\rangle$$
$$= \frac{1}{d} \sum_{\alpha, ijkl} \overline{U}^\alpha_{ij} \psi_l |\Psi_\alpha\rangle \langle ij|l\rangle |kk\rangle$$
$$= \frac{1}{d} \sum_{\alpha, ijk} \overline{U}^\alpha_{ij} \psi_i |\Psi_\alpha\rangle |j\rangle$$
$$= \frac{1}{d} \sum_\alpha |\Psi_\alpha\rangle (U^\alpha)^\dagger |\psi\rangle ,$$

where the last line is easily checked.